CHAPTER 11

11.1 First, the decomposition is implemented as

$e_2 = -0.4/0.8 = -0.5$
$f_2 = 0.8 - (-0.5)(-0.4) = 0.6$
$e_3 = -0.4/0.6 = -0.66667$
$f_3 = 0.8 - (-0.66667)(-0.4) = 0.53333$

Transformed system is

$$\begin{bmatrix} 0.8 & -0.4 & 0 \\ -0.5 & 0.6 & -0.4 \\ 0 & -0.66667 & 0.53333 \end{bmatrix}$$

which is decomposed as

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0 & -0.66667 & 1 \end{bmatrix} \qquad [U] = \begin{bmatrix} 0.8 & -0.4 & 0 \\ 0 & 0.6 & -0.4 \\ 0 & 0 & 0.53333 \end{bmatrix}$$

The right hand side becomes

$r_1 = 41$
$r_2 = 25 - (-0.5)(41) = 45.5$
$r_3 = 105 - (-0.66667)45.5 = 135.3333$

which can be used in conjunction with the $[U]$ matrix to perform back substitution and obtain the solution

$x_3 = 135.3333/0.53333 = 253.75$
$x_2 = (45.5 - (-0.4)253.75)/0.6 = 245$
$x_1 = (41 - (-0.4)245)/0.8 = 173.75$

11.2 As in Example 11.1, the *LU* decomposition is

$$[L] = \begin{bmatrix} 1 & & & \\ -0.49 & 1 & & \\ & -0.645 & 1 & \\ & & -0.717 & 1 \end{bmatrix} \qquad [U] = \begin{bmatrix} 2.04 & -1 & & \\ & 1.550 & -1 & \\ & & 1.395 & -1 \\ & & & 1.323 \end{bmatrix}$$

To compute the first column of the inverse

$$[L]\{D\} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Solving this gives

$$\{D\} = \begin{Bmatrix} 1 \\ 0.490196 \\ 0.316296 \\ 0.226775 \end{Bmatrix}$$

Back substitution, $[U]\{X\} = \{D\}$, can then be implemented to give to first column of the inverse

$$\{X\} = \begin{Bmatrix} 0.755841 \\ 0.541916 \\ 0.349667 \\ 0.171406 \end{Bmatrix}$$

For the second column

$$[L]\{D\} = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix}$$

which leads to

$$\{X\} = \begin{Bmatrix} 0.541916 \\ 1.105509 \\ 0.713322 \\ 0.349667 \end{Bmatrix}$$

For the third column

$$[L]\{D\} = \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix}$$

which leads to

$$\{X\} = \begin{Bmatrix} 0.349667 \\ 0.713322 \\ 1.105509 \\ 0.541916 \end{Bmatrix}$$

For the fourth column

$$[L]\{D\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix}$$

which leads to

$$\{X\} = \begin{Bmatrix} 0.171406 \\ 0.349667 \\ 0.541916 \\ 0.755841 \end{Bmatrix}$$

Therefore, the matrix inverse is

$$[A]^{-1} = \begin{bmatrix} 0.755841 & 0.541916 & 0.349667 & 0.171406 \\ 0.541916 & 1.105509 & 0.713322 & 0.349667 \\ 0.349667 & 0.713322 & 1.105509 & 0.541916 \\ 0.171406 & 0.349667 & 0.541916 & 0.755841 \end{bmatrix}$$

11.3 First, the decomposition is implemented as

e_2 = –0.020875/2.01475 = –0.01036
f_2 = 2.014534
e_3 = –0.01036
f_3 = 2.014534
e_4 = –0.01036
f_4 = 2.014534

Transformed system is

$$\begin{bmatrix} 2.01475 & -0.02875 & & \\ -0.01036 & 2.014534 & -0.02875 & \\ & -0.01036 & 2.014534 & -0.02875 \\ & & -0.01036 & 2.014534 \end{bmatrix}$$

which is decomposed as

$$[L] = \begin{bmatrix} 1 & & & \\ -0.01036 & 1 & & \\ & -0.01036 & 1 & \\ & & -0.01036 & 1 \end{bmatrix}$$

$$[U] = \begin{bmatrix} 2.01475 & -0.02875 & & \\ & 2.014534 & -0.02875 & \\ & & 2.014534 & -0.02875 \\ & & & 2.014534 \end{bmatrix}$$

Forward substitution yields

r_1 = 4.175
r_2 = 0.043258
r_3 = 0.000448

$r_4 = 2.087505$

Back substitution

$x_4 = 1.036222$
$x_3 = 0.01096$
$x_2 = 0.021586$
$x_1 = 2.072441$

11.4 We can use MATLAB to verify the results of Example 11.2,

```
>> L=[2.4495 0 0;6.1237 4.1833 0;22.454 20.916 6.1106]

L =
    2.4495         0         0
    6.1237    4.1833         0
   22.4540   20.9160    6.1106

>> L*L'

ans =
    6.0001   15.0000   55.0011
   15.0000   54.9997  224.9995
   55.0011  224.9995  979.0006
```

11.5

$$l_{11} = \sqrt{8} = 2.828427$$

$$l_{21} = \frac{20}{2.828427} = 7.071068$$

$$l_{22} = \sqrt{80 - 7.071068^2} = 5.477226$$

$$l_{31} = \frac{15}{2.828427} = 5.303301$$

$$l_{32} = \frac{50 - 7.071068(5.303301)}{5.477226} = 2.282177$$

$$l_{33} = \sqrt{60 - 5.303301^2 - 2.282177^2} = 5.163978$$

Thus, the Cholesky decomposition is

$$[L] = \begin{bmatrix} 2.828427 & & \\ 7.071068 & 5.477226 & \\ 5.303301 & 2.282177 & 5.163978 \end{bmatrix}$$

11.6

$$l_{11} = \sqrt{6} = 2.44949$$

$$l_{21} = \frac{15}{2.44949} = 6.123724$$

$$l_{22} = \sqrt{55 - 6.123724^2} = 4.1833$$

$$l_{31} = \frac{55}{2.44949} = 22.45366$$

$$l_{32} = \frac{225 - 6.123724(22.45366)}{4.1833} = 20.9165$$

$$l_{33} = \sqrt{979 - 22.45366^2 - 20.9165^2} = 6.110101$$

Thus, the Cholesky decomposition is

$$[L] = \begin{bmatrix} 2.44949 & & \\ 6.123724 & 4.1833 & \\ 22.45366 & 20.9165 & 6.110101 \end{bmatrix}$$

The solution can then be generated by first using forward substitution to modify the right-hand-side vector,

$$[L]\{D\} = \{B\}$$

which can be solved for

$$\{D\} = \begin{Bmatrix} 62.29869 \\ 48.78923 \\ 11.36915 \end{Bmatrix}$$

Then, we can use back substitution to determine the final solution,

$$[L]^T\{X\} = \{D\}$$

which can be solved for

$$\{D\} = \begin{Bmatrix} 2.478571 \\ 2.359286 \\ 1.860714 \end{Bmatrix}$$

11.7 (a) The first iteration can be implemented as

$$x_1 = \frac{41 + 0.4x_2}{0.8} = \frac{41 + 0.4(0)}{0.8} = 51.25$$

$$x_2 = \frac{25 + 0.4x_1 + 0.4x_3}{0.8} = \frac{25 + 0.4(51.25) + 0.4(0)}{0.8} = 56.875$$

$$x_3 = \frac{105 + 0.4x_2}{0.8} = \frac{105 + 0.4(56.875)}{0.8} = 159.6875$$

Second iteration:

$$x_1 = \frac{41 + 0.4(56.875)}{0.8} = 79.6875$$

$$x_2 = \frac{25 + 0.4(79.6875) + 0.4(159.6875)}{0.8} = 150.9375$$

$$x_3 = \frac{105 + 0.4(150.9375)}{0.8} = 206.7188$$

The error estimates can be computed as

$$\varepsilon_{a,1} = \left|\frac{79.6875 - 51.25}{79.6875}\right| \times 100\% = 35.69\%$$

$$\varepsilon_{a,2} = \left|\frac{150.9375 - 56.875}{150.9375}\right| \times 100\% = 62.32\%$$

$$\varepsilon_{a,3} = \left|\frac{206.7188 - 159.6875}{206.7188}\right| \times 100\% = 22.75\%$$

The remainder of the calculation proceeds until all the errors fall below the stopping criterion of 5%. The entire computation can be summarized as

iteration	unknown	value	ε_a	maximum ε_a
1	x_1	51.25	100.00%	
	x_2	56.875	100.00%	
	x_3	159.6875	100.00%	100.00%
2	x_1	79.6875	35.69%	
	x_2	150.9375	62.32%	
	x_3	206.7188	22.75%	62.32%
3	x_1	126.7188	37.11%	
	x_2	197.9688	23.76%	
	x_3	230.2344	10.21%	37.11%
4	x_1	150.2344	15.65%	
	x_2	221.4844	10.62%	

	x_3	241.9922	4.86%	15.65%
5	x_1	161.9922	7.26%	
	x_2	233.2422	5.04%	
	x_3	247.8711	2.37%	7.26%
6	x_1	167.8711	3.50%	
	x_2	239.1211	2.46%	
	x_3	250.8105	1.17%	3.50%

Thus, after 6 iterations, the maximum error is 3.5% and we arrive at the result: x_1 = 167.8711, x_2 = 239.1211 and x_3 = 250.8105.

(b) The same computation can be developed with relaxation where λ = 1.2.

First iteration:

$$x_1 = \frac{41 + 0.4x_2}{0.8} = \frac{41 + 0.4(0)}{0.8} = 51.25$$

Relaxation yields: $x_1 = 1.2(51.25) - 0.2(0) = 61.5$

$$x_2 = \frac{25 + 0.4x_1 + 0.4x_3}{0.8} = \frac{25 + 0.4(61.5) + 0.4(0)}{0.8} = 62$$

Relaxation yields: $x_2 = 1.2(62) - 0.2(0) = 74.4$

$$x_3 = \frac{105 + 0.4x_2}{0.8} = \frac{105 + 0.4(74.4)}{0.8} = 168.45$$

Relaxation yields: $x_3 = 1.2(168.45) - 0.2(0) = 202.14$

Second iteration:

$$x_1 = \frac{41 + 0.4(74.4)}{0.8} = 88.45$$

Relaxation yields: $x_1 = 1.2(88.45) - 0.2(61.5) = 93.84$

$$x_2 = \frac{25 + 0.4(93.84) + 0.4(202.14)}{0.8} = 179.24$$

Relaxation yields: $x_2 = 1.2(179.24) - 0.2(74.4) = 200.208$

$$x_3 = \frac{105 + 0.4(200.208)}{0.8} = 231.354$$

Relaxation yields: $x_3 = 1.2(231.354) - 0.2(202.14) = 237.1968$

The error estimates can be computed as

$$\varepsilon_{a,1} = \left|\frac{93.84 - 61.5}{93.84}\right| \times 100\% = 34.46\%$$

$$\varepsilon_{a,2} = \left|\frac{200.208 - 74.4}{200.208}\right| \times 100\% = 62.84\%$$

$$\varepsilon_{a,3} = \left|\frac{237.1968 - 202.14}{237.1968}\right| \times 100\% = 14.78\%$$

The remainder of the calculation proceeds until all the errors fall below the stopping criterion of 5%. The entire computation can be summarized as

iteration	unknown	value	relaxation	ε_a	maximum ε_a
1	x_1	51.25	61.5	100.00%	
	x_2	62	74.4	100.00%	
	x_3	168.45	202.14	100.00%	100.000%
2	x_1	88.45	93.84	34.46%	
	x_2	179.24	200.208	62.84%	
	x_3	231.354	237.1968	14.78%	62.839%
3	x_1	151.354	162.8568	42.38%	
	x_2	231.2768	237.49056	15.70%	
	x_3	249.99528	252.55498	6.08%	42.379%
4	x_1	169.99528	171.42298	5.00%	
	x_2	243.23898	244.38866	2.82%	
	x_3	253.44433	253.6222	0.42%	4.997%

Thus, relaxation speeds up convergence. After 6 iterations, the maximum error is 4.997% and we arrive at the result: x_1 = 171.423, x_2 = 244.389 and x_3 = 253.622.

11.8 The first iteration can be implemented as

$$c_1 = \frac{3800 + 3c_2 + c_3}{15} = \frac{3800 + 3(0) + 0}{15} = 253.3333$$

$$c_2 = \frac{1200 + 3c_1 + 6c_3}{18} = \frac{1200 + 3(253.3333) + 6(0)}{18} = 108.8889$$

$$c_3 = \frac{2350 + 4c_1 + c_2}{12} = \frac{2350 + 4(253.3333) + 108.8889}{12} = 289.3519$$

Second iteration:

$$c_1 = \frac{3800 + 3c_2 + c_3}{15} = \frac{3800 + 3(108.8889) + 289.3519}{15} = 294.4012$$

$$c_2 = \frac{1200 + 3c_1 + 6c_3}{18} = \frac{1200 + 3(294.4012) + 6(289.3519)}{18} = 212.1842$$

$$c_3 = \frac{2350 + 4c_1 + c_2}{12} = \frac{2350 + 4(294.4012) + 212.1842}{12} = 311.6491$$

The error estimates can be computed as

$$\varepsilon_{a,1} = \left|\frac{294.4012 - 253.3333}{294.4012}\right| \times 100\% = 13.95\%$$

$$\varepsilon_{a,2} = \left|\frac{212.1842 - 108.8889}{212.1842}\right| \times 100\% = 48.68\%$$

$$\varepsilon_{a,3} = \left|\frac{311.6491 - 289.3519}{311.6491}\right| \times 100\% = 7.15\%$$

The remainder of the calculation proceeds until all the errors fall below the stopping criterion of 5%. The entire computation can be summarized as

iteration	unknown	value	ε_a	maximum ε_a
1	c_1	253.3333	100.00%	
	c_2	108.8889	100.00%	
	c_3	289.3519	100.00%	100.00%
2	c_1	294.4012	13.95%	
	c_2	212.1842	48.68%	
	c_3	311.6491	7.15%	48.68%
3	c_1	316.5468	7.00%	
	c_2	223.3075	4.98%	
	c_3	319.9579	2.60%	7.00%
4	c_1	319.3254	0.87%	
	c_2	226.5402	1.43%	
	c_3	321.1535	0.37%	1.43%

Thus, after 4 iterations, the maximum error is 1.43% and we arrive at the result: c_1 = 319.3254, c_2 = 226.5402 and c_3 = 321.1535.

11.9 The first iteration can be implemented as

$$c_1 = \frac{3800 + 3c_2 + c_3}{15} = \frac{3800 + 3(0) + 0}{15} = 253.3333$$

$$c_2 = \frac{1200 + 3c_1 + 6c_3}{18} = \frac{1200 + 3(0) + 6(0)}{18} = 66.6667$$

$$c_3 = \frac{2350 + 4c_1 + c_2}{12} = \frac{2350 + 4(0) + 0}{12} = 195.8333$$

Second iteration:

$$c_1 = \frac{3800 + 3c_2 + c_3}{15} = \frac{3800 + 3(66.6667) + 195.8333}{15} = 279.7222$$

$$c_2 = \frac{1200 + 3c_1 + 6c_3}{18} = \frac{1200 + 3(253.3333) + 6(195.8333)}{18} = 174.1667$$

$$c_3 = \frac{2350 + 4c_1 + c_2}{12} = \frac{2350 + 4(253.3333) + 66.6667}{12} = 285.8333$$

The error estimates can be computed as

$$\varepsilon_{a,1} = \left| \frac{279.7222 - 253.3333}{279.7222} \right| \times 100\% = 9.43\%$$

$$\varepsilon_{a,2} = \left| \frac{174.1667 - 66.6667}{174.1667} \right| \times 100\% = 61.72\%$$

$$\varepsilon_{a,3} = \left| \frac{285.8333 - 195.8333}{285.8333} \right| \times 100\% = 31.49\%$$

The remainder of the calculation proceeds until all the errors fall below the stopping criterion of 5%. The entire computation can be summarized as

iteration	unknown	value	ε_a	maximum ε_a
1	c_1	253.3333	100.00%	
	c_2	66.66667	100.00%	
	c_3	195.8333	100.00%	100.00%
2	c_1	279.7222	9.43%	
	c_2	174.1667	61.72%	
	c_3	285.8333	31.49%	61.72%
3	c_1	307.2222	8.95%	
	c_2	208.5648	16.49%	
	c_3	303.588	5.85%	16.49%
4	c_1	315.2855	2.56%	
	c_2	219.0664	4.79%	
	c_3	315.6211	3.81%	4.79%

Thus, after 4 iterations, the maximum error is 4.79% and we arrive at the result: c_1 = 315.5402, c_2 = 219.0664 and c_3 = 315.6211.

11.10 The first iteration can be implemented as

$$x_1 = \frac{27 - 2x_2 + x_3}{10} = \frac{27 - 2(0) + 0}{10} = 2.7$$

$$x_2 = \frac{-61.5 + 3x_1 - 2x_3}{-6} = \frac{-61.5 + 3(2.7) - 2(0)}{-6} = 8.9$$

$$x_3 = \frac{-21.5 - x_1 - x_2}{5} = \frac{-21.5 - (2.7) - 8.9}{5} = -6.62$$

Second iteration:

$$x_1 = \frac{27 - 2(8.9) - 6.62}{10} = 0.258$$

$$x_2 = \frac{-61.5 + 3(0.258) - 2(-6.62)}{-6} = 7.914333$$

$$x_3 = \frac{-21.5 - (0.258) - 7.914333}{5} = -5.934467$$

The error estimates can be computed as

$$\varepsilon_{a,1} = \left| \frac{0.258 - 2.7}{0.258} \right| \times 100\% = 947\%$$

$$\varepsilon_{a,2} = \left| \frac{7.914333 - 8.9}{7.914333} \right| \times 100\% = 12.45\%$$

$$\varepsilon_{a,3} = \left| \frac{-5.934467 - (-6.62)}{-5.934467} \right| \times 100\% = 11.55\%$$

The remainder of the calculation proceeds until all the errors fall below the stopping criterion of 5%. The entire computation can be summarized as

iteration	unknown	value	ε_a	maximum ε_a
1	x_1	2.7	100.00%	
	x_2	8.9	100.00%	
	x_3	-6.62	100.00%	100%
2	x_1	0.258	946.51%	
	x_2	7.914333	12.45%	
	x_3	-5.93447	11.55%	946%

3	x_1	0.523687	50.73%	
	x_2	8.010001	1.19%	
	x_3	-6.00674	1.20%	50.73%
4	x_1	0.497326	5.30%	
	x_2	7.999091	0.14%	
	x_3	-5.99928	0.12%	5.30%
5	x_1	0.500253	0.59%	
	x_2	8.000112	0.01%	
	x_3	-6.00007	0.01%	0.59%

Thus, after 5 iterations, the maximum error is 0.59% and we arrive at the result: x_1 = 0.500253, x_2 = 8.000112 and x_3 = –6.00007.

11.11 The equations should first be rearranged so that they are diagonally dominant,

$$6x_1 - x_2 - x_3 = 3$$

$$6x_1 + 9x_2 + x_3 = 40$$

$$-3x_1 + x_2 + 12x_3 = 50$$

Each can be solved for the unknown on the diagonal as

$$x_1 = \frac{3 + x_2 + x_3}{6}$$

$$x_2 = \frac{40 - 6x_1 - x_3}{9}$$

$$x_3 = \frac{50 + 3x_1 - x_2}{12}$$

(a) The first iteration can be implemented as

$$x_1 = \frac{3 + 0 + 0}{6} = 0.5$$

$$x_2 = \frac{40 - 6(0.5) - 0}{9} = 4.11111$$

$$x_3 = \frac{50 + 3(0.5) - 4.11111}{12} = 3.949074$$

Second iteration:

$$x_1 = \frac{3 + 4.11111 + 3.949074}{6} = 1.843364$$

$$x_2 = \frac{40 - 6(1.843364) - 3.949074}{9} = 2.776749$$

$$x_3 = \frac{50 + 3(1.843364) - 2.776749}{12} = 4.396112$$

The error estimates can be computed as

$$\varepsilon_{a,1} = \left| \frac{1.843364 - 0.5}{1.843364} \right| \times 100\% = 72.88\%$$

$$\varepsilon_{a,2} = \left| \frac{2.776749 - 4.11111}{2.776749} \right| \times 100\% = 48.05\%$$

$$\varepsilon_{a,3} = \left| \frac{4.396112 - 3.949074}{4.396112} \right| \times 100\% = 10.17\%$$

The remainder of the calculation proceeds until all the errors fall below the stopping criterion of 5%. The entire computation can be summarized as

iteration	unknown	value	ε_a	maximum ε_a
1	x_1	0.5	100.00%	
	x_2	4.111111	100.00%	
	x_3	3.949074	100.00%	100.00%
2	x_1	1.843364	72.88%	
	x_2	2.776749	48.05%	
	x_3	4.396112	10.17%	72.88%
3	x_1	1.695477	8.72%	
	x_2	2.82567	1.73%	
	x_3	4.355063	0.94%	8.72%
4	x_1	1.696789	0.08%	
	x_2	2.829356	0.13%	
	x_3	4.355084	0.00%	0.13%

Thus, after 4 iterations, the maximum error is 0.13% and we arrive at the result: x_1 = 1.696789, x_2 = 2.829356 and x_3 = 4.355084.

(b) First iteration: To start, assume $x_1 = x_2 = x_3 = 0$

$$x_1^{new} = \frac{3 + 0 + 0}{6} = 0.5$$

Apply relaxation

$$x_1 = 0.95(0.5) + (1 - 0.95)0 = 0.475$$

$$x_2^{new} = \frac{40 - 6(0.475) - 0}{9} = 4.12778$$

$$x_2 = 0.95(4.12778) + (1 - 0.95)0 = 3.92139$$

$$x_3^{new} = \frac{50 + 3(0.475) - 3.92139}{12} = 3.95863$$

$$x_3 = 0.95(3.95863) + (1 - 0.95)0 = 3.76070$$

Note that error estimates are not made on the first iteration, because all errors will be 100%.

Second iteration:

$$x_1^{new} = \frac{3 + 3.92139 + 3.76070}{6} = 1.78035$$

$$x_1 = 0.95(1.78035) + (1 - 0.95)(0.475) = 1.71508$$

At this point, an error estimate can be made

$$\varepsilon_{a,1} = \left| \frac{1.71508 - 0.475}{1.71508} \right| 100\% = 72.3\%$$

Because this error exceeds the stopping criterion, it will not be necessary to compute error estimates for the remainder of this iteration.

$$x_2^{new} = \frac{40 - 6(1.71508) - 3.76070}{9} = 2.88320$$

$$x_2 = 0.95(2.88320) + (1 - 0.95)3.92139 = 2.93511$$

$$x_3^{new} = \frac{50 + 3(1.71508) - 2.93511}{12} = 4.35084$$

$$x_3 = 0.95(4.35084) + (1 - 0.95)3.76070 = 4.32134$$

The computations can be continued for one more iteration. The entire calculation is summarized in the following table.

iteration	x_1	x_{1r}	ε_{a1}	x_2	x_{2r}	ε_{a2}	x_3	x_{3r}	ε_{a3}
1	0.50000	0.47500	100.0%	4.12778	3.92139	100.0%	3.95863	3.76070	100.0%
2	1.78035	1.71508	72.3%	2.88320	2.93511	33.6%	4.35084	4.32134	13.0%
3	1.70941	1.70969	0.3%	2.82450	2.83003	3.7%	4.35825	4.35641	0.8%

After 3 iterations, the approximate errors fall below the stopping criterion with the final result: x_1 = 1.70969, x_2 = 2.82450 and x_3 = 4.35641. Note that the exact solution is x_1 = 1.69737, x_2 = 2.82895 and x_3 = 4.35526

11.12 The equations must first be rearranged so that they are diagonally dominant

$$-8x_1 + x_2 - 2x_3 = -20$$

$$2x_1 - 6x_2 - x_3 = -38$$

$$-3x_1 - x_2 + 7x_3 = -34$$

(a) The first iteration can be implemented as

$$x_1 = \frac{-20 - x_2 + 2x_3}{-8} = \frac{-20 - 0 + 2(0)}{-8} = 2.5$$

$$x_2 = \frac{-38 - 2x_1 + x_3}{-6} = \frac{-38 - 2(2.5) + 0}{-6} = 7.166667$$

$$x_3 = \frac{-34 + 3x_1 + x_2}{7} = \frac{-34 + 3(2.5) + 7.166667}{7} = -2.761905$$

Second iteration:

$$x_1 = \frac{-20 - 7.166667 + 2(-2.761905)}{-8} = 4.08631$$

$$x_2 = \frac{-38 - 2x_1 + x_3}{-6} = \frac{-38 - 2(4.08631) + (-2.761905)}{-6} = 8.155754$$

$$x_3 = \frac{-34 + 3x_1 + x_2}{7} = \frac{-34 + 3(4.08631) + 8.155754}{7} = -1.94076$$

The error estimates can be computed as

$$\varepsilon_{a,1} = \left| \frac{4.08631 - 2.5}{4.08631} \right| \times 100\% = 38.82\%$$

$$\varepsilon_{a,2} = \left| \frac{8.155754 - 7.166667}{8.155754} \right| \times 100\% = 12.13\%$$

$$\varepsilon_{a,3} = \left| \frac{-1.94076 - (-2.761905)}{-1.94076} \right| \times 100\% = 42.31\%$$

The remainder of the calculation proceeds until all the errors fall below the stopping criterion of 5%. The entire computation can be summarized as

iteration	unknown	value	ε_a	maximum ε_a
0	x_1	0		
	x_2	0		
	x_3	0		
1	x_1	2.5	100.00%	
	x_2	7.166667	100.00%	
	x_3	-2.7619	100.00%	100.00%
2	x_1	4.08631	38.82%	
	x_2	8.155754	12.13%	
	x_3	-1.94076	42.31%	42.31%
3	x_1	4.004659	2.04%	
	x_2	7.99168	2.05%	
	x_3	-1.99919	2.92%	2.92%

Thus, after 3 iterations, the maximum error is 2.92% and we arrive at the result: x_1 = 4.004659, x_2 = 7.99168 and x_3 = –1.99919.

(b) The same computation can be developed with relaxation where λ = 1.2.

First iteration:

$$x_1 = \frac{-20 - x_2 + 2x_3}{-8} = \frac{-20 - 0 + 2(0)}{-8} = 2.5$$

Relaxation yields: $x_1 = 1.2(2.5) - 0.2(0) = 3$

$$x_2 = \frac{-38 - 2x_1 + x_3}{-6} = \frac{-38 - 2(3) + 0}{-6} = 7.333333$$

Relaxation yields: $x_2 = 1.2(7.333333) - 0.2(0) = 8.8$

$$x_3 = \frac{-34 + 3x_1 + x_2}{7} = \frac{-34 + 3(3) + 8.8}{7} = -2.3142857$$

Relaxation yields: $x_3 = 1.2(-2.3142857) - 0.2(0) = -2.7771429$

Second iteration:

$$x_1 = \frac{-20 - x_2 + 2x_3}{-8} = \frac{-20 - 8.8 + 2(-2.7771429)}{-8} = 4.2942857$$

Relaxation yields: $x_1 = 1.2(4.2942857) - 0.2(3) = 4.5531429$

$$x_2 = \frac{-38 - 2x_1 + x_3}{-6} = \frac{-38 - 2(4.5531429) - 2.7771429}{-6} = 8.3139048$$

Relaxation yields: $x_2 = 1.2(8.3139048) - 0.2(8.8) = 8.2166857$

$$x_3 = \frac{-34 + 3x_1 + x_2}{7} = \frac{-34 + 3(4.5531429) + 8.2166857}{7} = -1.7319837$$

Relaxation yields: $x_3 = 1.2(-1.7319837) - 0.2(-2.7771429) = -1.5229518$

The error estimates can be computed as

$$\varepsilon_{a,1} = \left| \frac{4.5531429 - 3}{4.5531429} \right| \times 100\% = 34.11\%$$

$$\varepsilon_{a,2} = \left| \frac{8.2166857 - 8.8}{8.2166857} \right| \times 100\% = 7.1\%$$

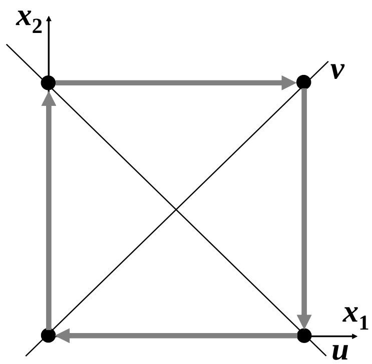
$$\varepsilon_{a,3} = \left| \frac{-1.5229518 - (-2.7771429)}{-1.5229518} \right| \times 100\% = 82.35\%$$

The remainder of the calculation proceeds until all the errors fall below the stopping criterion of 5%. The entire computation can be summarized as

iteration	unknown	value	relaxation	ε_a	maximum ε_a
1	x_1	2.5	3	100.00%	
	x_2	7.3333333	8.8	100.00%	
	x_3	-2.314286	-2.777143	100.00%	100.000%
2	x_1	4.2942857	4.5531429	34.11%	
	x_2	8.3139048	8.2166857	7.10%	
	x_3	-1.731984	-1.522952	82.35%	82.353%
3	x_1	3.9078237	3.7787598	20.49%	
	x_2	7.8467453	7.7727572	5.71%	
	x_3	-2.12728	-2.248146	32.26%	32.257%
4	x_1	4.0336312	4.0846055	7.49%	
	x_2	8.0695595	8.12892	4.38%	
	x_3	-1.945323	-1.884759	19.28%	19.280%
5	x_1	3.9873047	3.9678445	2.94%	
	x_2	7.9700747	7.9383056	2.40%	
	x_3	-2.022594	-2.050162	8.07%	8.068%
6	x_1	4.0048286	4.0122254	1.11%	
	x_2	8.0124354	8.0272613	1.11%	
	x_3	-1.990866	-1.979007	3.60%	3.595%

Thus, relaxation actually seems to retard convergence. After 6 iterations, the maximum error is 3.595% and we arrive at the result: $x_1 = 4.0122254$, $x_2 = 8.0272613$ and $x_3 = -1.979007$.

11.13 As shown below, for slopes of 1 and –1 the Gauss-Seidel technique will neither converge nor diverge but will oscillate interminably.



11.14 As ordered, none of the sets will converge. However, if Set 1 and 2 are reordered so that they are diagonally dominant, they will converge on the solution of (1, 1, 1).

Set 1:
$$\begin{aligned} 9x + 3y + z &= 13 \\ 2x + 5y - z &= 6 \\ -6x \qquad + 8z &= 2 \end{aligned}$$

Set 2:
$$\begin{aligned} 4x + 2y - 2z &= 4 \\ x + 5y - z &= 5 \\ x + y + 6z &= 8 \end{aligned}$$

At face value, because it is not strictly diagonally dominant, Set 2 would seem to be divergent. However, since it is very close to being diagonally dominant, a solution can be obtained.

The third set is not diagonally dominant and will diverge for most orderings. However, the following arrangement will converge albeit at a very slow rate:

Set 3:
$$\begin{aligned} -3x + 4y + 5z &= 6 \\ 2y - z &= 1 \\ -2x + 2y - 3z &= -3 \end{aligned}$$

11.15 Using MATLAB:

(a) The results for the first system will come out as expected.

```
>> A=[1 4 9;4 9 16;9 16 25]
>> B=[14 29 50]'
>> x=A\B

x =
    1.0000
    1.0000
    1.0000
```

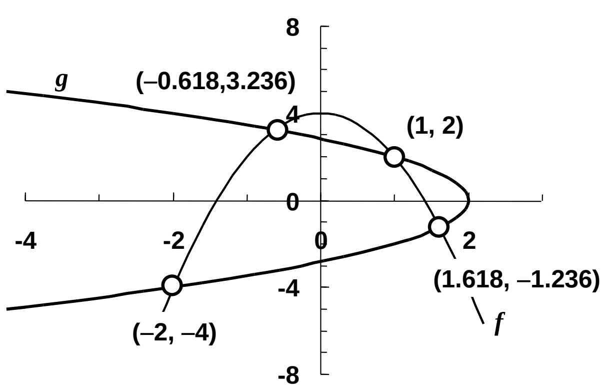
```
>> inv(A)

ans =
    3.8750   -5.5000    2.1250
   -5.5000    7.0000   -2.5000
    2.1250   -2.5000    0.8750

>> cond(A,inf)

ans =
  750.0000
```

(b) However, for the 4×4 system, the ill-conditioned nature of the matrix yields poor results:

```
>> A=[1 4 9 16;4 9 16 25;9 16 25 36;16 25 36 49];
>> B=[30 54 86 126]';
>> x=A\B
Warning: Matrix is close to singular or badly scaled.
         Results may be inaccurate. RCOND = 3.037487e-019.

x =
    0.5496
    2.3513
   -0.3513
    1.4504

>> cond(A,inf)
Warning: Matrix is close to singular or badly scaled.
         Results may be inaccurate. RCOND = 3.037487e-019.
> In cond at 48

ans =
  3.2922e+018
```

Note that using other software such as Excel yields similar results. For example, the condition number computed with Excel is 5×10^{17}.

11.16 (a) As shown, there are 4 roots, one in each quadrant.



(b) It might be expected that if an initial guess was within a quadrant, the result would be the root in the quadrant. However a sample of initial guesses spanning the range yield the following roots:

6	(-2, -4)	(-0.618,3.236)	(-0.618,3.236)	(1,2)	(-0.618,3.236)
3	(-0.618,3.236)	(-0.618,3.236)	(-0.618,3.236)	(1,2)	(-0.618,3.236)
0	(1,2)	(1.618, -1.236)	(1.618, -1.236)	(1.618, -1.236)	(1.618, -1.236)
-3	(-2, -4)	(-2, -4)	(1.618, -1.236)	(1.618, -1.236)	(1.618, -1.236)
-6	(-2, -4)	(-2, -4)	(-2, -4)	(1.618, -1.236)	(-2, -4)
	-6	-3	0	3	6

We have highlighted the guesses that converge to the roots in their quadrants. Although some follow the pattern, others jump to roots that are far away. For example, the guess of (–6, 0) jumps to the root in the first quadrant.

This underscores the notion that root location techniques are highly sensitive to initial guesses and that open methods like the Solver can locate roots that are not in the vicinity of the initial guesses.

11.17 Define the quantity of transistors, resistors, and computer chips as x_1, x_2 and x_3. The system equations can then be defined as

$$4x_1 + 3x_2 + 2x_3 = 960$$
$$x_1 + 3x_2 + x_3 = 510$$
$$2x_1 + x_2 + 3x_3 = 610$$

The solution can be implemented in Excel as shown below:

	A	B	C	D
1	A:			B:
2	4	3	2	960
3	1	3	1	510
4	2	1	3	610
5				
6	AI:			X:
7	0.421053	-0.36842	-0.15789	120
8	-0.05263	0.421053	-0.10526	100
9	-0.26316	0.105263	0.473684	90

The following view shows the formulas that are employed to determine the inverse in cells A7:C9 and the solution in cells D7:D9.

	A	B	C	D
1	A:			B:
2	4	3	2	960
3	1	3	1	510
4	2	1	3	610
5				
6	AI:			X:
7	=MINVERSE(A2:C4)	=MINVERSE(A2:C4)	=MINVERSE(A2:C4)	=MMULT(A7:C9,D2:D4)
8	=MINVERSE(A2:C4)	=MINVERSE(A2:C4)	=MINVERSE(A2:C4)	=MMULT(A7:C9,D2:D4)
9	=MINVERSE(A2:C4)	=MINVERSE(A2:C4)	=MINVERSE(A2:C4)	=MMULT(A7:C9,D2:D4)

Here is the same solution generated in MATLAB:

```
>> A=[4 3 2;1 3 1;2 1 3];
>> B=[960 510 610]';
>> x=A\B

x =
   120
   100
    90
```

In both cases, the answer is $x_1 = 120$, $x_2 = 100$, and $x_3 = 90$

11.18 The spectral condition number can be evaluated as

```
>> A = hilb(10);
>> N = cond(A)

N =
  1.6025e+013
```

The digits of precision that could be lost due to ill-conditioning can be calculated as

```
>> c = log10(N)

c =
   13.2048
```

Thus, about 13 digits could be suspect. A right-hand side vector can be developed corresponding to a solution of ones:

```
>> b=[sum(A(1,:)); sum(A(2,:)); sum(A(3,:)); sum(A(4,:)); sum(A(5,:));
sum(A(6,:)); sum(A(7,:)); sum(A(8,:)); sum(A(9,:)); sum(A(10,:))]

b =
    2.9290
    2.0199
    1.6032
    1.3468
    1.1682
    1.0349
    0.9307
    0.8467
    0.7773
    0.7188
```

The solution can then be generated by left division

```
>> x = A\b

x =
    1.0000
    1.0000
```

```
    1.0000
    1.0000
    0.9999
    1.0003
    0.9995
    1.0005
    0.9997
    1.0001
```

The maximum and mean errors can be computed as

```
>> e=max(abs(x-1))

e =
  5.3822e-004

>> e=mean(abs(x-1))

e =
  1.8662e-004
```

Thus, some of the results are accurate to only about 3 to 4 significant digits. Because MATLAB represents numbers to 15 significant digits, this means that about 11 to 12 digits are suspect.

11.19 First, the Vandermonde matrix can be set up

```
>> x1 = 4;x2=2;x3=7;x4=10;x5=3;x6=5;
>> A = [x1^5 x1^4 x1^3 x1^2 x1 1;x2^5 x2^4 x2^3 x2^2 x2 1;x3^5 x3^4
x3^3 x3^2 x3 1;x4^5 x4^4 x4^3 x4^2 x4 1;x5^5 x5^4 x5^3 x5^2 x5 1;x6^5
x6^4 x6^3 x6^2 x6 1]

A =
        1024         256          64          16           4           1
          32          16           8           4           2           1
       16807        2401         343          49           7           1
      100000       10000        1000         100          10           1
         243          81          27           9           3           1
        3125         625         125          25           5           1
```

The spectral condition number can be evaluated as

```
>> N = cond(A)

N =
  1.4492e+007
```

The digits of precision that could be lost due to ill-conditioning can be calculated as

```
>> c = log10(N)

c =
    7.1611
```

Thus, about 7 digits might be suspect. A right-hand side vector can be developed corresponding to a solution of ones:

```
>> b=[sum(A(1,:));sum(A(2,:));sum(A(3,:));sum(A(4,:));sum(A(5,:));
sum(A(6,:))]

b =
        1365
          63
       19608
      111111
         364
        3906
```

The solution can then be generated by left division

```
>> format long
>> x=A\b

x =
   1.00000000000000
   0.99999999999991
   1.00000000000075
   0.99999999999703
   1.00000000000542
   0.99999999999630
```

The maximum and mean errors can be computed as

```
>> e = max(abs(x-1))

e =
    5.420774940034789e-012

>> e = mean(abs(x-1))

e =
    2.154110223528960e-012
```

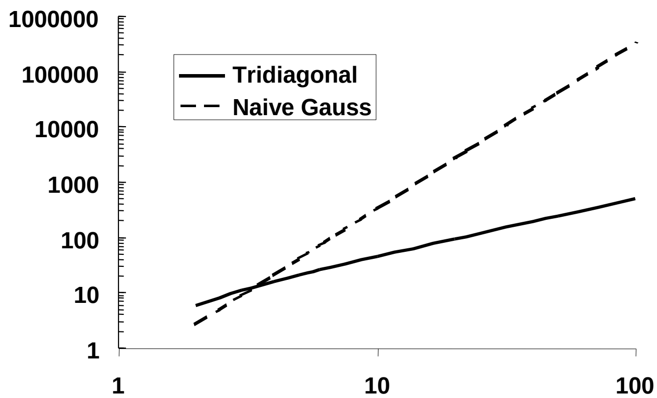
Some of the results are accurate to about 12 significant digits. Because MATLAB represents numbers to about 15 significant digits, this means that about 3 digits are suspect. Thus, for this case, the condition number tends to exaggerate the impact of ill-conditioning.

11.20 The flop counts for the tridiagonal algorithm in Fig. 11.2 can be determined as

```
                                                  mult/div     add/subt
Sub Decomp(e, f, g, n)
Dim k As Integer
For k = 2 To n
  e(k) = e(k) / f(k - 1)                         '(n - 1)
  f(k) = f(k) - e(k) * g(k - 1)                  '(n - 1)      (n - 1)
Next k
End Sub

Sub Substitute(e, f, g, r, n, x)
```

```
Dim k As Integer
For k = 2 To n
  r(k) = r(k) - e(k) * r(k - 1)          '(n - 1)      (n - 1)
Next k
x(n) = r(n) / f(n)                       '   1
For k = n - 1 To 1 Step -1
  x(k) = (r(k) - g(k) * x(k + 1)) / f(k) '2(n - 1)     (n - 1)
Next k
End Sub

Sum =                                    5(n-1) + 1    (3n - 3)
```

The multiply/divides and add/subtracts can be summed to yield $8n - 7$ as opposed to $n^3/3$ for naive Gauss elimination. Therefore, a tridiagonal solver is well worth using.



11.21 Here is a VBA macro to obtain a solution for a tridiagonal system using the Thomas algorithm. It is set up to duplicate the results of Example 11.1.

```
Option Explicit

Sub TriDiag()
Dim i As Integer, n As Integer
Dim e(10) As Double, f(10) As Double, g(10) As Double
Dim r(10) As Double, x(10) As Double
n = 4
e(2) = -1: e(3) = -1: e(4) = -1
f(1) = 2.04: f(2) = 2.04: f(3) = 2.04: f(4) = 2.04
g(1) = -1: g(2) = -1: g(3) = -1
r(1) = 40.8: r(2) = 0.8: r(3) = 0.8: r(4) = 200.8
Call Thomas(e, f, g, r, n, x)
For i = 1 To n
  MsgBox x(i)
Next i
End Sub

Sub Thomas(e, f, g, r, n, x)
Call Decomp(e, f, g, n)
Call Substitute(e, f, g, r, n, x)
End Sub

Sub Decomp(e, f, g, n)
Dim k As Integer
For k = 2 To n
  e(k) = e(k) / f(k - 1)
```

```
  f(k) = f(k) - e(k) * g(k - 1)
Next k
End Sub

Sub Substitute(e, f, g, r, n, x)
Dim k As Integer
For k = 2 To n
  r(k) = r(k) - e(k) * r(k - 1)
Next k
x(n) = r(n) / f(n)
For k = n - 1 To 1 Step -1
  x(k) = (r(k) - g(k) * x(k + 1)) / f(k)
Next k
End Sub
```

11.22 Here is a VBA macro to obtain a solution of a symmetric system with Cholesky decomposition. It is set up to duplicate the results of Example 11.2.

```
Option Explicit

Sub TestChol()
Dim i As Integer, j As Integer
Dim n As Integer
Dim a(10, 10) As Double
n = 3
a(1, 1) = 6: a(1, 2) = 15: a(1, 3) = 55
a(2, 1) = 15: a(2, 2) = 55: a(2, 3) = 225
a(3, 1) = 55: a(3, 2) = 225: a(3, 3) = 979
Call Cholesky(a, n)
'output results to worksheet
Sheets("Sheet1").Select
Range("a3").Select
For i = 1 To n
  For j = 1 To n
    ActiveCell.Value = a(i, j)
    ActiveCell.Offset(0, 1).Select
  Next j
  ActiveCell.Offset(1, -n).Select
Next i
Range("a3").Select
End Sub

Sub Cholesky(a, n)
Dim i As Integer, j As Integer, k As Integer
Dim sum As Double
For k = 1 To n
  For i = 1 To k - 1
    sum = 0
    For j = 1 To i - 1
      sum = sum + a(i, j) * a(k, j)
    Next j
    a(k, i) = (a(k, i) - sum) / a(i, i)
  Next i
  sum = 0
  For j = 1 To k - 1
    sum = sum + a(k, j) ^ 2
  Next j
  a(k, k) = Sqr(a(k, k) - sum)
Next k
End Sub
```

11.23 Here is a VBA macro to obtain a solution of a linear diagonally-dominant system with the Gauss-Seidel method. It is set up to duplicate the results of Example 11.3.

```
Option Explicit

Sub Gausseid()
Dim n As Integer, imax As Integer, i As Integer
Dim a(3, 3) As Double, b(3) As Double, x(3) As Double
Dim es As Double, lambda As Double
n = 3
a(1, 1) = 3: a(1, 2) = -0.1: a(1, 3) = -0.2
a(2, 1) = 0.1: a(2, 2) = 7: a(2, 3) = -0.3
a(3, 1) = 0.3: a(3, 2) = -0.2: a(3, 3) = 10
b(1) = 7.85: b(2) = -19.3: b(3) = 71.4
es = 0.1
imax = 20
lambda = 1#
Call Gseid(a, b, n, x, imax, es, lambda)
For i = 1 To n
  MsgBox x(i)
Next i
End Sub

Sub Gseid(a, b, n, x, imax, es, lambda)
Dim i As Integer, j As Integer, iter As Integer, sentinel As Integer
Dim dummy As Double, sum As Double, ea As Double, old As Double
For i = 1 To n
  dummy = a(i, i)
  For j = 1 To n
    a(i, j) = a(i, j) / dummy
  Next j
  b(i) = b(i) / dummy
Next i
For i = 1 To n
  sum = b(i)
  For j = 1 To n
    If i <> j Then sum = sum - a(i, j) * x(j)
  Next j
  x(i) = sum
Next i
iter = 1
Do
  sentinel = 1
  For i = 1 To n
    old = x(i)
    sum = b(i)
    For j = 1 To n
      If i <> j Then sum = sum - a(i, j) * x(j)
    Next j
    x(i) = lambda * sum + (1# - lambda) * old
    If sentinel = 1 And x(i) <> 0 Then
      ea = Abs((x(i) - old) / x(i)) * 100
      If ea > es Then sentinel = 0
    End If
  Next i
  iter = iter + 1
  If sentinel = 1 Or iter >= imax Then Exit Do
Loop
End Sub
```