CHAPTER 13

13.1 (a) The function can be differentiated to give

$$f'(x) = -2x + 8$$

This function can be set equal to zero and solved for $x = 8/2 = 4$. The derivative can be differentiated to give the second derivative

$$f''(x) = -2$$

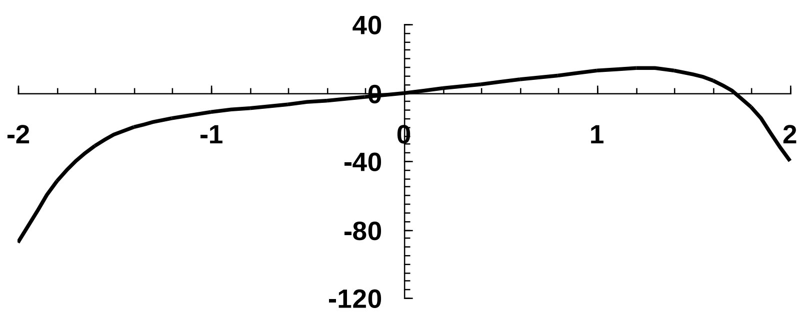
Because this is negative, it indicates that the function has a maximum at $x = 4$.

(b) Using Eq. 13.7

$x_0 = 0$ $f(x_0) = -12$
$x_1 = 2$ $f(x_1) = 0$
$x_2 = 6$ $f(x_2) = 0$

$$x_3 = \frac{-12(4-36) + 0(36-0) + 0(0-4)}{2(-12)(2-6) + 2(0)(6-0) + 2(0)(0-2)} = 4$$

13.2 (a) The function can be plotted



(b) The function can be differentiated twice to give

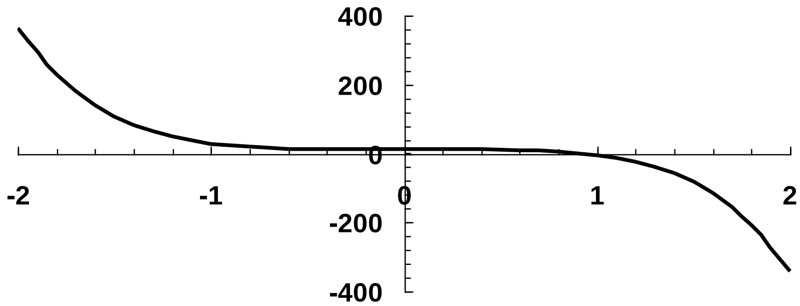
$$f''(x) = -45x^4 - 24x^2$$

Thus, the second derivative will always be negative and hence the function is concave for all values of x.

(c) Differentiating the function and setting the result equal to zero results in the following roots problem to locate the maximum

$$f'(x) = 0 = -9x^5 - 8x^3 + 12$$

A plot of this function can be developed



A technique such as bisection can be employed to determine the root. Here are the first few iterations:

iteration	x_l	x_u	x_r	$f(x_l)$	$f(x_r)$	$f(x_l)\times f(x_r)$	ε_a
1	0.00000	2.00000	1.00000	12	-5	-60.0000	
2	0.00000	1.00000	0.50000	12	10.71875	128.6250	100.00%
3	0.50000	1.00000	0.75000	10.71875	6.489258	69.5567	33.33%
4	0.75000	1.00000	0.87500	6.489258	2.024445	13.1371	14.29%
5	0.87500	1.00000	0.93750	2.024445	-1.10956	-2.2463	6.67%

The approach can be continued to yield a result of x = 0.91692.

13.3 First, the golden ratio can be used to create the interior points,

$$d = \frac{\sqrt{5}-1}{2}(2-0) = 1.2361$$
$$x_1 = 0 + 1.2361 = 1.2361$$
$$x_2 = 2 - 1.2361 = 0.7639$$

The function can be evaluated at the interior points

$$f(x_2) = f(0.7639) = 8.1879$$
$$f(x_1) = f(1.2361) = 4.8142$$

Because $f(x_2) > f(x_1)$, the maximum is in the interval defined by x_l, x_2, and x_1.where x_2 is the optimum. The error at this point can be computed as

$$\varepsilon_a = (1-0.61803)\left|\frac{2-0}{0.7639}\right| \times 100\% = 100\%$$

For the second iteration, x_l = 0 and x_u = 1.2361. The former x_2 value becomes the new x_1, that is, x_1 = 0.7639 and $f(x_1)$ = 8.1879. The new values of d and x_2 can be computed as

$$d = \frac{\sqrt{5}-1}{2}(1.2361 - 0) = 0.7639$$
$$x_2 = 1.2361 - 0.7639 = 0.4721$$

The function evaluation at $f(x_2)$ = 5.5496. Since this value is less than the function value at x_1, the maximum is in the interval prescribed by x_2, x_1 and x_u. The process can be repeated and all three iterations summarized as

i	x_l	$f(x_l)$	x_2	$f(x_2)$	x_1	$f(x_1)$	x_u	$f(x_u)$	d	x_{opt}	ε_a
1	0.0000	0.0000	0.7639	8.1879	1.2361	4.8142	2.0000	-104.0000	1.2361	0.7639	100.00%
2	0.0000	0.0000	0.4721	5.5496	0.7639	8.1879	1.2361	4.8142	0.7639	0.7639	61.80%
3	0.4721	5.5496	0.7639	8.1879	0.9443	8.6778	1.2361	4.8142	0.4721	0.9443	30.90%

13.4 First, the function values at the initial values can be evaluated

$$f(x_0) = f(0) = 0$$
$$f(x_1) = f(1) = 8.5$$
$$f(x_2) = f(2) = -104$$

and substituted into Eq. (13.7) to give,

$$x_3 = \frac{0(1^2 - 2^2) + 8.5(2^2 - 0^2) + (-104)(0^2 - 1^2)}{2(0)(1-2) + 2(8.5)(2-0) + 2(-104)(0-1)} = 0.570248$$

which has a function value of f(0.570248) = 6.5799. Because the function value for the new point is lower than for the intermediate point (x_1) and the new x value is to the left of the intermediate point, the lower guess (x_0) is discarded. Therefore, for the next iteration,

$$f(x_0) = f(0.570248) = 6.6799$$
$$f(x_1) = f(1) = 8.5$$
$$f(x_2) = f(2) = -104$$

which can be substituted into Eq. (13.7) to give x_3 = 0.812431, which has a function value of f(0.812431) = 8.446523. At this point, an approximate error can be computed as

$$\varepsilon_a = \left|\frac{0.81243 - 0.570248}{0.81243}\right| \times 100\% = 29.81\%$$

The process can be repeated, with the results tabulated below:

i	x_0	$f(x_0)$	x_1	$f(x_1)$	x_2	$f(x_2)$	x_3	$f(x_3)$	ε_a
1	0.00000	0.00000	1.00000	8.50000	2.0000	-104	0.57025	6.57991	
2	0.57025	6.57991	1.00000	8.50000	2.0000	-104	0.81243	8.44652	29.81%
3	0.81243	8.44652	1.00000	8.50000	2.0000	-104	0.90772	8.69575	10.50%

Thus, after 3 iterations, the result is converging on the true value of $f(x)$ = 8.69793 at x = 0.91692.

13.5 The first and second derivatives of the function can be evaluated as

$$f'(x) = -9x^5 - 8x^3 + 12$$
$$f''(x) = -45x^4 - 24x^2$$

which can be substituted into Eq. (13.8) to give

$$x_{i+1} = x_i - \frac{-9x_i^5 - 8x_i^3 + 12}{-45x_i^4 - 24x_i^2}$$

Substituting the initial guess yields

$$x_{i+1} = 2 - \frac{-9(2^5) - 8(2^3) + 12}{-45(2^4) - 24(2^2)} = 2 - \frac{-340}{-816} = 1.583333$$

which has a function value of –17.2029. The second iteration gives

$$x_{i+1} = 1.583333 - \frac{-9(1.583333^5) - 8(1.583333^3) + 12}{-45(1.583333^4) - 24(1.583333^2)} = 1.583333 - \frac{-109.313}{-342.981} = 1.26462$$

which has a function value of 3.924617. At this point, an approximate error can be computed as

$$\varepsilon_a = \left|\frac{1.26462 - 1.583333}{1.26462}\right| \times 100\% = 26.316\%$$

The process can be repeated, with the results tabulated below:

i	x	$f(x)$	$f'(x)$	$f''(x)$	ε_a
0	2	-104	-340	-816	
1	1.583333	-17.2029	-109.313	-342.981	26.316%
2	1.26462	3.924617	-33.2898	-153.476	25.202%
3	1.047716	8.178616	-8.56281	-80.5683	20.703%

Thus, within five iterations, the result is converging on the true value of $f(x)$ = 8.69793 at x = 0.91692.

13.6 Golden section search is inefficient, but always converges if x_l and x_u bracket the maximum or minimum of a unimodal function.

Quadratic interpolation can be programmed as either a bracketing or as an open method. For the former, convergence is guaranteed if the initial guesses bracket the maximum or minimum of a unimodal function. However, as mentioned at the top of p. 351, it may sometimes converge slowly. If it is programmed as an open method, it may converge rapidly for well-behaved functions and good initial values. Otherwise, it may diverge. It also has the disadvantage that three initial guesses are required.

Newton's method may converge rapidly for well-behaved functions and good initial values. Otherwise, it may diverge. It also has the disadvantage that both the first and second derivatives must be determined.

13.7 (a) First, the golden ratio can be used to create the interior points,

$$d = \frac{\sqrt{5}-1}{2}(4-(-2)) = 3.7082$$
$$x_1 = -2 + 3.7082 = 1.7082$$
$$x_2 = 4 - 3.7082 = 0.2918$$

The function can be evaluated at the interior points

$$f(x_2) = f(0.2918) = 1.04156$$
$$f(x_1) = f(1.7082) = 5.00750$$

Because $f(x_1) > f(x_2)$, the maximum is in the interval defined by x_2, x_1 and x_u where x_1 is the optimum. The error at this point can be computed as

$$\varepsilon_a = (1-0.61803)\left|\frac{4-(-2)}{1.7082}\right| \times 100\% = 134.16\%$$

The process can be repeated and all the iterations summarized as

i	x_l	$f(x_l)$	x_2	$f(x_2)$	x_1	$f(x_1)$	x_u	$f(x_u)$	d	x_{opt}	ε_a
1	-2.0000	-29.6000	0.2918	1.0416	1.7082	5.0075	4.0000	-12.8000	3.7082	1.7082	134.16%
2	0.2918	1.0416	1.7082	5.0075	2.5836	5.6474	4.0000	-12.8000	2.2918	2.5836	54.82%
3	1.7082	5.0075	2.5836	5.6474	3.1246	2.9361	4.0000	-12.8000	1.4164	2.5836	33.88%
4	1.7082	5.0075	2.2492	5.8672	2.5836	5.6474	3.1246	2.9361	0.8754	2.2492	24.05%
5	1.7082	5.0075	2.0426	5.6648	2.2492	5.8672	2.5836	5.6474	0.5410	2.2492	14.87%
6	2.0426	5.6648	2.2492	5.8672	2.3769	5.8770	2.5836	5.6474	0.3344	2.3769	8.69%
7	2.2492	5.8672	2.3769	5.8770	2.4559	5.8287	2.5836	5.6474	0.2067	2.3769	5.37%
8	2.2492	5.8672	2.3282	5.8853	2.3769	5.8770	2.4559	5.8287	0.1277	2.3282	3.39%
9	2.2492	5.8672	2.2980	5.8828	2.3282	5.8853	2.3769	5.8770	0.0789	2.3282	2.10%
10	2.2980	5.8828	2.3282	5.8853	2.3468	5.8840	2.3769	5.8770	0.0488	2.3282	1.30%
11	2.2980	5.8828	2.3166	5.8850	2.3282	5.8853	2.3468	5.8840	0.0301	2.3282	0.80%

(b) First, the function values at the initial values can be evaluated

$$f(x_0) = f(1.75) = 5.1051$$
$$f(x_1) = f(2) = 5.6$$
$$f(x_2) = f(2.5) = 5.7813$$

and substituted into Eq. (13.7) to give,

$$x_3 = \frac{5.1051(2^2 - 2.5^2) + 5.6(2.5^2 - 1.75^2) + 5.7813(1.75^2 - 2^2)}{2(5.1051)(2 - 2.5) + 2(5.6)(2.5 - 1.75) + 2(5.7813)(1.75 - 2)} = 2.3341$$

which has a function value of $f(2.3341) = 5.8852$. Because the function value for the new point is higher than for the intermediate point (x_1) and the new x value is to the right of the intermediate point, the lower guess (x_0) is discarded. Therefore, for the next iteration,

$$f(x_0) = f(2) = 5.6$$
$$f(x_1) = f(2.3341) = 5.8852$$
$$f(x_2) = f(2.5) = 5.7813$$

which can be substituted into Eq. (13.7) to give $x_3 = 2.3112$, which has a function value of $f(2.3112) = 5.8846$. At this point, an approximate error can be computed as

$$\varepsilon_a = \left| \frac{2.3112 - 2.3341}{2.3112} \right| \times 100\% = 0.99\%$$

The process can be repeated, with the results tabulated below:

i	x_0	$f(x_0)$	x_1	$f(x_1)$	x_2	$f(x_2)$	x_3	$f(x_3)$	ε_a
1	1.7500	5.1051	2.0000	5.6000	2.5000	5.7813	2.3341	5.8852	
2	2.0000	5.6000	2.3341	5.8852	2.5000	5.7813	2.3112	5.8846	0.99%
3	2.3112	5.8846	2.3341	5.8852	2.5000	5.7813	2.3260	5.8853	0.64%
4	2.3112	5.8846	2.3260	5.8853	2.3341	5.8852	2.3263	5.8853	0.01%

Thus, after 4 iterations, the result is converging rapidly on the true value of $f(x) = 5.8853$ at $x = 2.3263$.

(c) The first and second derivatives of the function can be evaluated as

$$f'(x) = 4 - 3.6x + 3.6x^2 - 1.2x^3$$
$$f''(x) = -3.6 + 7.2x - 3.6x^2$$

which can be substituted into Eq. (13.8) to give

$$x_{i+1} = x_i - \frac{4 - 3.6x_i + 3.6x_i^2 - 1.2x_i^3}{-3.6 + 7.2x_i - 3.6x_i^2} = 3 - \frac{-6.8}{-14.4} = 2.5278$$

which has a function value of 5.7434. The second iteration gives 2.3517, which has a function value of 5.8833. At this point, an approximate error can be computed as $\varepsilon_a = 18.681\%$. The process can be repeated, with the results tabulated below:

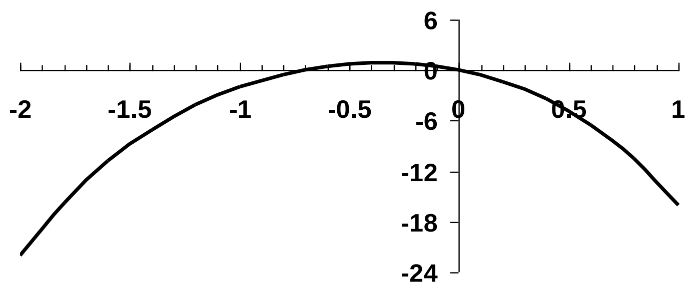
i	*x*	*f*(*x*)	*f'*(*x*)	*f"*(*x*)	ε_a
0	3.0000	3.9000	-6.8000	-14.4000	
1	2.5278	5.7434	-1.4792	-8.4028	18.681%
2	2.3517	5.8833	-0.1639	-6.5779	7.485%
3	2.3268	5.8853	-0.0030	-6.3377	1.071%
4	2.3264	5.8853	0.0000	-6.3332	0.020%

Thus, within four iterations, the result is converging on the true value of $f(x) = 5.8853$ at $x = 2.3264$.

13.8 The function can be differentiated twice to give

$$f'(x) = -4x^3 - 6x^2 - 16x - 5$$
$$f''(x) = -12x^2 - 12x - 16$$

which is negative for $-2 \le x \le 1$. This suggests that an optimum in the interval would be a maximum. A graph of the original function shows a maximum at about $x = -0.35$.



13.9 (a) First, the golden ratio can be used to create the interior points,

$$d = \frac{\sqrt{5}-1}{2}(1-(-2)) = 1.8541$$
$$x_1 = -2 + 1.8541 = -0.1459$$
$$x_2 = 1 - 1.8541 = -0.8541$$

The function can be evaluated at the interior points

$$f(x_2) = f(-0.8541) = -0.8514$$
$$f(x_1) = f(-0.1459) = 0.5650$$

Because $f(x_1) > f(x_2)$, the maximum is in the interval defined by x_2, x_1 and x_u where x_1 is the optimum. The error at this point can be computed as

$$\varepsilon_a = (1 - 0.61803)\left|\frac{1-(-2)}{-0.1459}\right| \times 100\% = 785.41\%$$

The process can be repeated and all the iterations summarized as

i	x_l	$f(x_l)$	x_2	$f(x_2)$	x_1	$f(x_1)$	x_u	$f(x_u)$	d	x_{opt}	ε_a
1	-2	-22	-0.8541	-0.851	-0.1459	0.565	1	-16.000	1.8541	-0.1459	785.41%
2	-0.8541	-0.851	-0.1459	0.565	0.2918	-2.197	1	-16.000	1.1459	-0.1459	485.41%
3	-0.8541	-0.851	-0.4164	0.809	-0.1459	0.565	0.2918	-2.197	0.7082	-0.4164	105.11%
4	-0.8541	-0.851	-0.5836	0.475	-0.4164	0.809	-0.1459	0.565	0.4377	-0.4164	64.96%
5	-0.5836	0.475	-0.4164	0.809	-0.3131	0.833	-0.1459	0.565	0.2705	-0.3131	53.40%
6	-0.4164	0.809	-0.3131	0.833	-0.2492	0.776	-0.1459	0.565	0.1672	-0.3131	33.00%
7	-0.4164	0.809	-0.3525	0.841	-0.3131	0.833	-0.2492	0.776	0.1033	-0.3525	18.11%
8	-0.4164	0.809	-0.3769	0.835	-0.3525	0.841	-0.3131	0.833	0.0639	-0.3525	11.19%
9	-0.3769	0.835	-0.3525	0.841	-0.3375	0.840	-0.3131	0.833	0.0395	-0.3525	6.92%
10	-0.3769	0.835	-0.3619	0.839	-0.3525	0.841	-0.3375	0.840	0.0244	-0.3525	4.28%
11	-0.3619	0.839	-0.3525	0.841	-0.3468	0.841	-0.3375	0.840	0.0151	-0.3468	2.69%
12	-0.3525	0.841	-0.3468	0.841	-0.3432	0.841	-0.3375	0.840	0.0093	-0.3468	1.66%
13	-0.3525	0.841	-0.3490	0.841	-0.3468	0.841	-0.3432	0.841	0.0058	-0.3468	1.03%
14	-0.3490	0.841	-0.3468	0.841	-0.3454	0.841	-0.3432	0.841	0.0036	-0.3468	0.63%

(b) First, the function values at the initial values can be evaluated

$$f(x_0) = f(-2) = -22$$
$$f(x_1) = f(-1) = -2$$
$$f(x_2) = f(1) = -16$$

and substituted into Eq. (13.7) to give,

$$x_3 = \frac{-22((-1)^2 - 1^2) + (-2)(1^2 - (-2)^2) + (-16)((-2)^2 - (-1)^2)}{2(-22)(-1-1) + 2(-2)(1-(-2)) + 2(-16)(-2-(-1))} = -0.38889$$

which has a function value of $f(-0.38889) = 0.829323$. Because the function value for the new point is higher than for the intermediate point (x_1) and the new x value is to the right of the intermediate point, the lower guess (x_0) is discarded. Therefore, for the next iteration,

$$f(x_0) = f(-1) = -2$$
$$f(x_1) = f(-0.38889) = 0.829323$$
$$f(x_2) = f(1) = -16$$

which can be substituted into Eq. (13.7) to give $x_3 = -0.41799$, which has a function value of $f(-0.41799) = 0.80776$. At this point, an approximate error can be computed as

$$\varepsilon_a = \left| \frac{-0.41799 - (-0.38889)}{-0.41799} \right| \times 100\% = 6.96\%$$

The process can be repeated, with the results tabulated below:

i	x_0	$f(x_0)$	x_1	$f(x_1)$	x_2	$f(x_2)$	x_3	$f(x_3)$	ε_a
1	-2	-22	-1.0000	-2	1.0000	-16	-0.3889	0.8293	
2	-1.0000	-2	-0.3889	0.8293	1.0000	-16	-0.4180	0.8078	6.96%
3	-0.4180	0.8078	-0.3889	0.8293	1.0000	-16	-0.3626	0.8392	15.28%
4	-0.3889	0.8293	-0.3626	0.8392	1.0000	-16	-0.3552	0.8404	2.08%

After 4 iterations, the result is converging on the true value of $f(x) = 0.8408$ at $x = -0.34725$.

(c) The first and second derivatives of the function can be evaluated as

$$f'(x) = -4x^3 - 6x^2 - 16x - 5$$
$$f''(x) = -12x^2 - 12x - 16$$

which can be substituted into Eq. (13.8) to give

$$x_{i+1} = x_i - \frac{-4x_i^3 - 6x_i^2 - 16x_i - 5}{-12x_i^2 - 12x_i - 16} = -1 - \frac{9}{-16} = -0.4375$$

which has a function value of 0.787094. The second iteration gives –0.34656, which has a function value of 0.840791. At this point, an approximate error can be computed as ε_a = 128.571%. The process can be repeated, with the results tabulated below:

i	x	$f(x)$	$f'(x)$	$f''(x)$	ε_a
0	-1	-2	9	-16	
1	-0.4375	0.787094	1.186523	-13.0469	128.571%
2	-0.34656	0.840791	-0.00921	-13.2825	26.242%
3	-0.34725	0.840794	-8.8E-07	-13.28	0.200%

Thus, within three iterations, the result is converging on the true value of $f(x) = 0.840794$ at $x = -0.34725$.

13.10 First, the function values at the initial values can be evaluated

$$f(x_0) = f(0.1) = 30.2$$
$$f(x_1) = f(0.5) = 7$$
$$f(x_2) = f(5) = 10.6$$

and substituted into Eq. (13.7) to give,

$$x_3 = \frac{30.2(0.5^2 - 5^2) + 7(5^2 - 0.1^2) + 10.6(0.1^2 - 0.5^2)}{2(30.2)(0.5 - 5) + 2(7)(5 - 0.1) + 2(10.6)(0.1 - 0.5)} = 2.7167$$

which has a function value of $f(2.7167) = 6.5376$. Because the function value for the new point is lower than for the intermediate point (x_1) and the new x value is to the right of the intermediate point, the lower guess (x_0) is discarded. Therefore, for the next iteration,

$f(x_0) = f(0.5) = 7$
$f(x_1) = f(2.7167) = 6.5376$
$f(x_2) = f(5) = 10.6$

which can be substituted into Eq. (13.7) to give x_3 = 1.8444, which has a function value of *f*(1.8444) = 5.3154. At this point, an approximate error can be computed as

$$\varepsilon_a = \left| \frac{1.8444 - 2.7167}{1.8444} \right| \times 100\% = 47.29\%$$

The process can be repeated, with the results tabulated below:

i	x_0	$f(x_0)$	x_1	$f(x_1)$	x_2	$f(x_2)$	x_3	$f(x_3)$	ε_a
1	0.1	30.2	0.5000	7.0000	5.0000	10.6000	2.7167	6.5376	
2	0.5000	7	2.7167	6.5376	5.0000	10.6000	1.8444	5.3154	47.29%
3	0.5000	7	1.8444	5.3154	2.7167	6.5376	1.6954	5.1603	8.79%
4	0.5000	7	1.6954	5.1603	1.8444	5.3154	1.4987	4.9992	13.12%
5	0.5000	7	1.4987	4.9992	1.6954	5.1603	1.4236	4.9545	5.28%
6	0.5000	7	1.4236	4.9545	1.4987	4.9992	1.3556	4.9242	5.02%
7	0.5000	7	1.3556	4.9242	1.4236	4.9545	1.3179	4.9122	2.85%
8	0.5000	7	1.3179	4.9122	1.3556	4.9242	1.2890	4.9054	2.25%
9	0.5000	7	1.2890	4.9054	1.3179	4.9122	1.2703	4.9023	1.47%
10	0.5000	7	1.2703	4.9023	1.2890	4.9054	1.2568	4.9006	1.08%

Thus, after 10 iterations, the result is converging very slowly on the true value of *f*(*x*) = 4.8990 at *x* = 1.2247.

13.11 Differentiating the function and setting the result equal to zero results in the following roots problem to locate the minimum

$$f'(x) = 6 + 10x + 9x^2 + 16x^3$$

Bisection can be employed to determine the root. Here are the first few iterations:

iteration	x_l	x_u	x_r	$f(x_l)$	$f(x_r)$	$f(x_l) \times f(x_r)$	ε_a
1	-2.0000	1.0000	-0.5000	-106.000	1.2500	-132.500	300.00%
2	-2.0000	-0.5000	-1.2500	-106.000	-23.6875	2510.875	60.00%
3	-1.2500	-0.5000	-0.8750	-23.6875	-6.5781	155.819	42.86%
4	-0.8750	-0.5000	-0.6875	-6.5781	-1.8203	11.974	27.27%
5	-0.6875	-0.5000	-0.5938	-1.8203	-0.1138	0.207	15.79%
6	-0.5938	-0.5000	-0.5469	-0.1138	0.6060	-0.0689	8.57%
7	-0.5938	-0.5469	-0.5703	-0.1138	0.2562	-0.0291	4.11%
8	-0.5938	-0.5703	-0.5820	-0.1138	0.0738	-0.0084	2.01%
9	-0.5938	-0.5820	-0.5879	-0.1138	-0.0193	0.0022	1.00%
10	-0.5879	-0.5820	-0.5850	-0.0193	0.0274	-0.0005	0.50%

The approach can be continued to yield a result of *x* = –0.5867.

13.12 (a) The first and second derivatives of the function can be evaluated as

$$f'(x) = 6 + 10x + 9x^2 + 16x^3$$
$$f''(x) = 10 + 18x + 48x^2$$

which can be substituted into Eq. (13.8) to give

$$x_{i+1} = x_i - \frac{6 + 10x_i + 9x_i^2 + 16x_i^3}{10 + 18x_i + 48x_i^2} = -1 - \frac{-11}{40} = -0.725$$

which has a function value of 1.24. The second iteration gives –0.60703, which has a function value of 1.07233. At this point, an approximate error can be computed as ε_a = 37.931%. The process can be repeated, with the results tabulated below:

i	*x*	*f*(*x*)	*f'*(*x*)	*f"*(*x*)	ε_a
0	-1	3	-11	40	
1	-0.725	1.24002	-2.61663	22.180	37.931%
2	-0.60703	1.07233	-0.33280	16.76067	19.434%
3	-0.58717	1.06897	-0.00781	15.97990	3.382%
4	-0.58668	1.06897	-4.6E-06	15.96115	0.083%

Thus, within four iterations, the stopping criterion is met and the result is converging on the true value of *f*(*x*) = 1.06897 at *x* = –0.58668.

(b) The finite difference approximations of the derivatives can be computed as

$$f'(x) = \frac{3.1120 - 2.8920}{-0.01} = -11.001$$

$$f''(x) = \frac{3.1120 - 2(3) + 2.8920}{(-0.01)^2} = 40.001$$

which can be substituted into Eq. (13.8) to give

$$x_{i+1} = x_i - \frac{6 + 10x_i + 9x_i^2 + 16x_i^3}{10 + 18x_i + 48x_i^2} = -1 - \frac{-11.001}{40.001} = -0.725$$

which has a function value of 1.2399. The second iteration gives –0.6070, which has a function value of 1.0723. At this point, an approximate error can be computed as ε_a = 37.936%. The process can be repeated, with the results tabulated below:

i	x_i	$f(x_i)$	δx_i	$x_i-\delta x_i$	$f(x_i-\delta x_i)$	$x_i+\delta x_i$	$f(x_i+\delta x_i)$	$f'(x_i)$	$f''(x_i)$	ε_a
0	-1	3	-0.01	-0.99	2.8920	-1.0100	3.1120	-11.001	40.001	
1	-0.7250	1.2399	-0.00725	-0.7177	1.2216	-0.7322	1.2595	-2.616	22.179	37.94%
2	-0.6070	1.0723	-0.00607	-0.6009	1.0706	-0.6131	1.0746	-0.333	16.760	19.43%

3	-0.5872	1.0690	-0.00587	-0.5813	1.0692	-0.5930	1.0693	-0.008	15.980	3.38%
4	-0.5867	1.0690	-0.00587	-0.5808	1.0692	-0.5925	1.0692	-4.1E-06	15.961	0.081%

Thus, within four iterations, the stopping criterion is met and the result is converging on the true value of $f(x) = 1.06897$ at $x = -0.58668$.

13.13 Because of multiple local minima and maxima, there is no really simple means to test whether a single maximum occurs within an interval without actually performing a search. However, if we assume that the function has one maximum and no minima within the interval, a check can be included. Here is a VBA program to implement the Golden section search algorithm for maximization and solve Example 13.1.

```
Option Explicit

Sub GoldMax()
Dim ier As Integer
Dim xlow As Double, xhigh As Double
Dim xopt As Double, fopt As Double
xlow = 0
xhigh = 4
Call GoldMx(xlow, xhigh, xopt, fopt, ier)
If ier = 0 Then
  MsgBox "xopt = " & xopt
  MsgBox "f(xopt) = " & fopt
Else
  MsgBox "Does not appear to be maximum in [xl, xu]"
End If
End Sub

Sub GoldMx(xlow, xhigh, xopt, fopt, ier)
Dim iter As Integer, maxit As Integer, ea As Double, es As Double
Dim xL As Double, xU As Double, d As Double, x1 As Double
Dim x2 As Double, f1 As Double, f2 As Double
Const R As Double = (5 ^ 0.5 - 1) / 2
ier = 0
maxit = 50
es = 0.001
xL = xlow
xU = xhigh
iter = 1
d = R * (xU - xL)
x1 = xL + d
x2 = xU - d
f1 = f(x1)
f2 = f(x2)
If f1 > f2 Then
  xopt = x1
  fopt = f1
Else
  xopt = x2
  fopt = f2
End If
If fopt > f(xL) And fopt > f(xU) Then
  Do
    d = R * d
    If f1 > f2 Then
      xL = x2
      x2 = x1
```

```
      x1 = xL + d
      f2 = f1
      f1 = f(x1)
    Else
      xU = x1
      x1 = x2
      x2 = xU - d
      f1 = f2
      f2 = f(x2)
    End If
    iter = iter + 1
    If f1 > f2 Then
      xopt = x1
      fopt = f1
    Else
      xopt = x2
      fopt = f2
    End If
    If xopt <> 0 Then ea = (1 - R) * Abs((xU - xL) / xopt) * 100
    If ea <= es Or iter >= maxit Then Exit Do
  Loop
Else
  ier = 1
End If
End Sub

Function f(x)
f = 2 * Sin(x) - x ^ 2 / 10
End Function
```

13.14 The easiest way to set up a maximization algorithm so that it can do minimization is to realize that minimizing a function is the same as maximizing its negative. Therefore, the following algorithm written in VBA minimizes or maximizes depending on the value of a user input variable, ind, where ind = –1 and 1 correspond to minimization and maximization, respectively. It is set up to solve the minimization described in Prob. 13.10.

```
Option Explicit

Sub GoldMinMax()
Dim ind As Integer       'Minimization (ind = -1); Maximization (ind = 1)
Dim xlow As Double, xhigh As Double
Dim xopt As Double, fopt As Double
xlow = 0.1
xhigh = 5
Call GoldMnMx(xlow, xhigh, -1, xopt, fopt)
MsgBox "xopt = " & xopt
MsgBox "f(xopt) = " & fopt
End Sub

Sub GoldMnMx(xlow, xhigh, ind, xopt, fopt)
Dim iter As Integer, maxit As Integer, ea As Double, es As Double
Dim xL As Double, xU As Double, d As Double, x1 As Double
Dim x2 As Double, f1 As Double, f2 As Double
Const R As Double = (5 ^ 0.5 - 1) / 2
maxit = 50
es = 0.001
xL = xlow
xU = xhigh
iter = 1
d = R * (xU - xL)
```

```
x1 = xL + d
x2 = xU - d
f1 = f(ind, x1)
f2 = f(ind, x2)
If f1 > f2 Then
  xopt = x1
  fopt = f1
Else
  xopt = x2
  fopt = f2
End If
Do
  d = R * d
  If f1 > f2 Then
    xL = x2
    x2 = x1
    x1 = xL + d
    f2 = f1
    f1 = f(ind, x1)
  Else
    xU = x1
    x1 = x2
    x2 = xU - d
    f1 = f2
    f2 = f(ind, x2)
  End If
  iter = iter + 1
  If f1 > f2 Then
    xopt = x1
    fopt = f1
  Else
    xopt = x2
    fopt = f2
  End If
  If xopt <> 0 Then ea = (1 - R) * Abs((xU - xL) / xopt) * 100
  If ea <= es Or iter >= maxit Then Exit Do
Loop
fopt = ind * fopt
End Sub

Function f(ind, x)
f = 2 * x + 3 / x  'place function to be evaluated here
f = ind * f
End Function
```

13.15 Because of multiple local minima and maxima, there is no really simple means to test whether a single maximum occurs within an interval without actually performing a search. However, if we assume that the function has one maximum and no minima within the interval, a check can be included. Here is a VBA program to implement the Quadratic Interpolation algorithm for maximization and solve Example 13.2.

```
Option Explicit

Sub QuadMax()
Dim ier As Integer
Dim xlow As Double, xhigh As Double
Dim xopt As Double, fopt As Double
xlow = 0
xhigh = 4
Call QuadMx(xlow, xhigh, xopt, fopt, ier)
```

```
If ier = 0 Then
  MsgBox "xopt = " & xopt
  MsgBox "f(xopt) = " & fopt
Else
  MsgBox "Does not appear to be maximum in [xl, xu]"
End If
End Sub

Sub QuadMx(xlow, xhigh, xopt, fopt, ier)
Dim iter As Integer, maxit As Integer, ea As Double, es As Double
Dim x0 As Double, x1 As Double, x2 As Double
Dim f0 As Double, f1 As Double, f2 As Double
Dim xoptOld As Double
ier = 0
maxit = 50
es = 0.0001
x0 = xlow
x2 = xhigh
x1 = (x0 + x2) / 2
f0 = f(x0)
f1 = f(x1)
f2 = f(x2)
If f1 > f0 Or f1 > f2 Then
  xoptOld = x1
  Do
    xopt = f0 * (x1 ^ 2 - x2 ^ 2) + f1 * (x2 ^ 2 - x0 ^ 2) + f2 * (x0 ^ 2 - x1 ^ 2)
    xopt = xopt / (2 * f0 * (x1 - x2) + 2 * f1 * (x2 - x0) + 2 * f2 * (x0 - x1))
    fopt = f(xopt)
    iter = iter + 1
    If xopt > x1 Then
      x0 = x1
      f0 = f1
      x1 = xopt
      f1 = fopt
    Else
      x2 = x1
      f2 = f1
      x1 = xopt
      f1 = fopt
    End If
    If xopt <> 0 Then ea = Abs((xopt - xoptOld) / xopt) * 100
    xoptOld = xopt
    If ea <= es Or iter >= maxit Then Exit Do
  Loop
Else
  ier = 1
End If
End Sub

Function f(x)
f = 2 * Sin(x) - x ^ 2 / 10
End Function
```

13.16 Here is a VBA program to implement the Newton-Raphson method for maximization. It is set up to duplicate the computation from Example 13.3.

```
Option Explicit

Sub NRMax()
Dim xguess As Double
Dim xopt As Double, fopt As Double
```

```
xguess = 2.5
Call NRMx(xguess, xopt, fopt)
MsgBox "xopt = " & xopt
MsgBox "f(xopt) = " & fopt
End Sub

Sub NRMx(xguess, xopt, fopt)
Dim iter As Integer, maxit As Integer, ea As Double, es As Double
Dim x0 As Double, x1 As Double, x2 As Double
Dim f0 As Double, f1 As Double, f2 As Double
Dim xoptOld As Double
maxit = 50
es = 0.01
Do
  xopt = xguess - df(xguess) / d2f(xguess)
  fopt = f(xopt)
  If xopt <> 0 Then ea = Abs((xopt - xguess) / xopt) * 100
  xguess = xopt
  If ea <= es Or iter >= maxit Then Exit Do
Loop
End Sub

Function f(x)
f = 2 * Sin(x) - x ^ 2 / 10
End Function

Function df(x)
df = 2 * Cos(x) - x / 5
End Function

Function d2f(x)
d2f = -2 * Sin(x) - 1 / 5
End Function
```

13.17 The first iteration of the golden-section search can be implemented as

$$d = \frac{\sqrt{5}-1}{2}(4-2) = 1.2361$$
$$x_1 = 2 + 1.2361 = 3.2361$$
$$x_2 = 4 - 1.2361 = 2.7639$$
$$f(x_2) = f(2.7639) = -6.1303$$
$$f(x_1) = f(3.2361) = -5.8317$$

Because $f(x_2) < f(x_1)$, the minimum is in the interval defined by x_l, x_2, and x_1 where x_2 is the optimum. The error at this point can be computed as

$$\varepsilon_a = (1 - 0.61803)\left|\frac{4-2}{2.7639}\right| \times 100\% = 27.64\%$$

The process can be repeated and all the iterations summarized as

i	x_l	$f(x_l)$	x_2	$f(x_2)$	x_1	$f(x_1)$	x_u	$f(x_u)$	d	x_{opt}	ε_a
1	2	-3.8608	2.7639	-6.1303	3.2361	-5.8317	4	-2.7867	1.2361	2.7639	27.64%
2	2	-3.8608	2.4721	-5.6358	2.7639	-6.1303	3.2361	-5.8317	0.7639	2.7639	17.08%

3	2.4721	-5.6358	2.7639	-6.1303	2.9443	-6.1776	3.2361	-5.8317	0.4721	2.9443	9.91%
4	2.7639	-6.1303	2.9443	-6.1776	3.0557	-6.1065	3.2361	-5.8317	0.2918	2.9443	6.13%

After four iterations, the process is converging on the true minimum at x = 2.8966 where the function has a value of $f(x) = -6.1847$.

13.18 The first iteration of the golden-section search can be implemented as

$$d = \frac{\sqrt{5}-1}{2}(60-0) = 37.0820$$
$$x_1 = 0 + 37.0820 = 37.0820$$
$$x_2 = 60 - 37.0820 = 22.9180$$
$$f(x_2) = f(22.9180) = 18.336$$
$$f(x_1) = f(37.0820) = 19.074$$

Because $f(x_1) > f(x_2)$, the maximum is in the interval defined by x_2, x_1, and x_u where x_1 is the optimum. The error at this point can be computed as

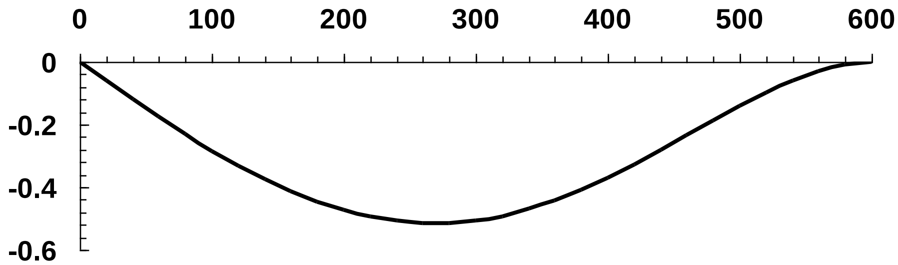
$$\varepsilon_a = (1-0.61803)\left|\frac{60-0}{37.0820}\right| \times 100\% = 61.80\%$$

The process can be repeated and all the iterations summarized as

i	x_l	$f(x_l)$	x_2	$f(x_2)$	x_1	$f(x_1)$	x_u	$f(x_u)$	d	x_{opt}	ε_a
1	0	1	22.9180	18.336	37.0820	19.074	60	4.126	37.0820	37.0820	61.80%
2	22.9180	18.336	37.0820	19.074	45.8359	15.719	60	4.126	22.9180	37.0820	38.20%
3	22.9180	18.336	31.6718	19.692	37.0820	19.074	45.8359	15.719	14.1641	31.6718	27.64%
4	22.9180	18.336	28.3282	19.518	31.6718	19.692	37.0820	19.074	8.7539	31.6718	17.08%
5	28.3282	19.518	31.6718	19.692	33.7384	19.587	37.0820	19.074	5.4102	31.6718	10.56%
6	28.3282	19.518	30.3947	19.675	31.6718	19.692	33.7384	19.587	3.3437	31.6718	6.52%
7	30.3947	19.675	31.6718	19.692	32.4612	19.671	33.7384	19.587	2.0665	31.6718	4.03%
8	30.3947	19.675	31.1840	19.693	31.6718	19.692	32.4612	19.671	1.2772	31.1840	2.53%
9	30.3947	19.675	30.8825	19.689	31.1840	19.693	31.6718	19.692	0.7893	31.1840	1.56%
10	30.8825	19.689	31.1840	19.693	31.3703	19.693	31.6718	19.692	0.4878	31.3703	0.96%

After ten iterations, the process falls below the stopping criterion and the result is converging on the true maximum at x = 31.3713 where the function has a value of y = 19.6934.

13.19 (a) A graph indicates the minimum at about x = 270.



(b) Golden section search,

$$d = \frac{\sqrt{5}-1}{2}(600-0) = 370.820$$
$$x_1 = 0 + 370.820 = 370.820$$
$$x_2 = 600 - 370.820 = 229.180$$
$$f(x_2) = f(229.180) = -0.5016$$
$$f(x_1) = f(370.820) = -0.4249$$

Because $f(x_2) < f(x_1)$, the minimum is in the interval defined by x_l, x_2, and x_1 where x_2 is the optimum. The error at this point can be computed as

$$\varepsilon_a = (1-0.61803)\left|\frac{600-0}{370.820}\right| \times 100\% = 100\%$$

The process can be repeated and all the iterations summarized as

i	x_l	$f(x_l)$	x_2	$f(x_2)$	x_1	$f(x_1)$	x_u	$f(x_u)$	d	x_{opt}	ε_a
1	0	0	229.180	-0.5016	370.820	-0.4249	600	0	370.820	229.1796	100.00%
2	0	0	141.641	-0.3789	229.180	-0.5016	370.820	-0.4249	229.180	229.1796	61.80%
3	141.641	-0.3789	229.180	-0.5016	283.282	-0.5132	370.820	-0.4249	141.641	283.2816	30.90%
4	229.180	-0.5016	283.282	-0.5132	316.718	-0.4944	370.820	-0.4249	87.539	283.2816	19.10%
5	229.180	-0.5016	262.616	-0.5149	283.282	-0.5132	316.718	-0.4944	54.102	262.6165	12.73%
6	229.180	-0.5016	249.845	-0.5121	262.616	-0.5149	283.282	-0.5132	33.437	262.6165	7.87%
7	249.845	-0.5121	262.616	-0.5149	270.510	-0.5151	283.282	-0.5132	20.665	270.5098	4.72%
8	262.616	-0.5149	270.510	-0.5151	275.388	-0.5147	283.282	-0.5132	12.772	270.5098	2.92%
9	262.616	-0.5149	267.495	-0.5152	270.510	-0.5151	275.388	-0.5147	7.893	267.4948	1.82%
10	262.616	-0.5149	265.631	-0.5151	267.495	-0.5152	270.510	-0.5151	4.878	267.4948	1.13%
11	265.631	-0.5151	267.495	-0.5152	268.646	-0.5152	270.510	-0.5151	3.015	268.6465	0.69%

After eleven iterations, the process falls below the stopping criterion and the result is converging on the true minimum at x = 268.3281 where the function has a value of y = –0.51519.

13.20 The velocity of a falling object with an initial velocity and first-order drag can be computed as

$$v = v_0 e^{-(c/m)t} + \frac{mg}{c}\left(1 - e^{-(c/m)t}\right)$$

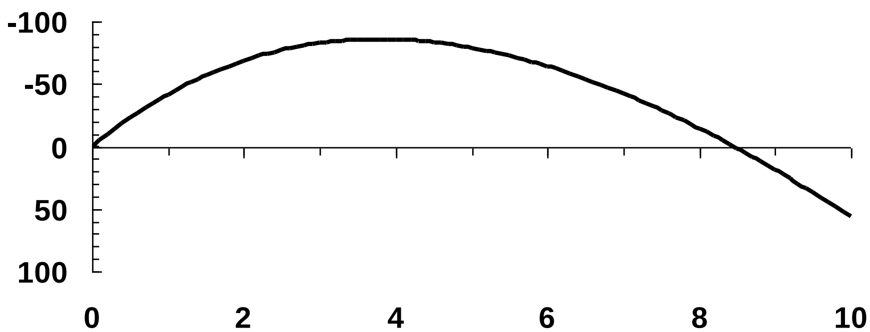
The vertical distance traveled can be determined by integration

$$z = z_0 + \int_0^t v_0 e^{-(c/m)t} + \frac{mg}{c}\left(1 - e^{-(c/m)t}\right) dt$$

where negative z is distance upwards. Assuming that $z_0 = 0$, evaluating the integral yields

$$z = \frac{mg}{c} t + \frac{m}{c}\left(v_0 - \frac{mg}{c}\right)\left(1 - e^{-(c/m)t}\right)$$

Therefore the solution to this problem amounts to determining the minimum of this function (since the most negative value of z corresponds to the maximum height). This function can be plotted using the given parameter values. As in the following graph (note that the ordinate values are plotted in reverse), the maximum height occurs after about 3.5 s and appears to be about 75 m.



Here is the result of using the golden-section search to determine the maximum height

i	x_l	$f(x_l)$	x_2	$f(x_2)$	x_1	$f(x_1)$	x_u	$f(x_u)$	d	x_{opt}	ε_a
1	0	0	1.9098	-66.733	3.0902	-83.278	5	-78.926	3.0902	3.0902	61.80%
2	1.9098	-66.733	3.0902	-83.278	3.8197	-85.731	5	-78.926	1.9098	3.8197	30.90%
3	3.0902	-83.278	3.8197	-85.731	4.2705	-84.612	5.0000	-78.926	1.1803	3.8197	19.10%
4	3.0902	-83.278	3.5410	-85.439	3.8197	-85.731	4.2705	-84.612	0.7295	3.8197	11.80%
5	3.5410	-85.439	3.8197	-85.731	3.9919	-85.531	4.2705	-84.612	0.4508	3.8197	7.29%
6	3.5410	-85.439	3.7132	-85.711	3.8197	-85.731	3.9919	-85.531	0.2786	3.8197	4.51%
7	3.7132	-85.711	3.8197	-85.731	3.8854	-85.688	3.9919	-85.531	0.1722	3.8197	2.79%
8	3.7132	-85.711	3.7790	-85.737	3.8197	-85.731	3.8854	-85.688	0.1064	3.7790	1.74%
9	3.7132	-85.711	3.7539	-85.732	3.7790	-85.737	3.8197	-85.731	0.0658	3.7790	1.08%
10	3.7539	-85.732	3.7790	-85.737	3.7945	-85.736	3.8197	-85.731	0.0407	3.7790	0.66%

After ten iterations, the process falls below a stopping of 1% criterion and the result is converging on the true minimum at x = 3.78588 where the function has a value of y = –85.7367. Thus, the maximum height is 74.811.

13.21 The inflection point corresponds to the point at which the derivative of the normal distribution is a minimum. The derivative can be evaluated as

$$\frac{dy}{dx} = -2xe^{-x^2}$$

Starting with initial guesses of x_l = 0 and x_u = 2, the golden-section search method can be implemented as

$$d = \frac{\sqrt{5}-1}{2}(2-0) = 1.2361$$
$$x_1 = 0 + 1.2361 = 1.2361$$
$$x_2 = 2 - 1.2361 = 0.7639$$
$$f(x_2) = f(0.7639) = -2(0.7639)e^{-(0.7639)^2} = -0.8524$$
$$f(x_1) = f(1.2361) = -2(1.2361)e^{-(1.2361)^2} = -0.5365$$

Because $f(x_2) < f(x_1)$, the minimum is in the interval defined by x_l, x_2, and x_1 where x_2 is the optimum. The error at this point can be computed as

$$\varepsilon_a = (1-0.61803)\left|\frac{2-0}{0.7639}\right| \times 100\% = 100\%$$

The process can be repeated and all the iterations summarized as

i	x_l	$f(x_l)$	x_2	$f(x_2)$	x_1	$f(x_1)$	x_u	$f(x_u)$	*d*	x_{opt}	ε_a
1	0	0.0000	0.7639	-0.8524	1.2361	-0.5365	2	-0.0733	1.2361	0.7639	100.00%
2	0	0.0000	0.4721	-0.7556	0.7639	-0.8524	1.2361	-0.5365	0.7639	0.7639	61.80%
3	0.4721	-0.7556	0.7639	-0.8524	0.9443	-0.7743	1.2361	-0.5365	0.4721	0.7639	38.20%
4	0.4721	-0.7556	0.6525	-0.8525	0.7639	-0.8524	0.9443	-0.7743	0.2918	0.6525	27.64%
5	0.4721	-0.7556	0.5836	-0.8303	0.6525	-0.8525	0.7639	-0.8524	0.1803	0.6525	17.08%
6	0.5836	-0.8303	0.6525	-0.8525	0.6950	-0.8575	0.7639	-0.8524	0.1115	0.6950	9.91%
7	0.6525	-0.8525	0.6950	-0.8575	0.7214	-0.8574	0.7639	-0.8524	0.0689	0.6950	6.13%
8	0.6525	-0.8525	0.6788	-0.8564	0.6950	-0.8575	0.7214	-0.8574	0.0426	0.6950	3.79%
9	0.6788	-0.8564	0.6950	-0.8575	0.7051	-0.8578	0.7214	-0.8574	0.0263	0.7051	2.31%
10	0.6950	-0.8575	0.7051	-0.8578	0.7113	-0.8577	0.7214	-0.8574	0.0163	0.7051	1.43%
11	0.6950	-0.8575	0.7013	-0.8577	0.7051	-0.8578	0.7113	-0.8577	0.0100	0.7051	0.88%

After eleven iterations, the process falls below a stopping of 1% criterion and the result is converging on the true minimum at x = 0.707107 where the function has a value of y = –0.85776.