CHAPTER 6

6.1 The function can be formulated as a fixed-point iteration as

$$x_{i+1} = 2\sin\left(\sqrt{x_i}\right)$$

Using an initial guess of $x_0 = 0.5$, the first iteration is

$$x_1 = 2\sin\left(\sqrt{0.5}\right) = 1.299274$$

$$\varepsilon_a = \left|\frac{1.299274 - 0.5}{1.299274}\right| \times 100\% = 61.517\%$$

The remaining iterations are summarized below. As can be seen, 7 iterations are required to drop below the specified stopping criterion of 0.01%.

i	*x*	ε_a	ratio
0	0.5		
1	1.299274	61.517%	
2	1.817148	28.499%	0.46327
3	1.950574	6.840%	0.24002
4	1.969743	0.973%	0.14227
5	1.972069	0.118%	0.12122
6	1.972344	0.014%	0.11832
7	1.972377	0.002%	0.11797

The table also includes a column showing the ratio of the relative errors between iterations:

$$\text{ratio} = \frac{\varepsilon_{a,i}}{\varepsilon_{a,i-1}}$$

As can be seen, after the first few iterations this ratio is converging on a constant value of about 0.118. Recall that the error of fixed-point iteration is

$$E_{t,i+1} = g'(\xi) E_{t,i}$$

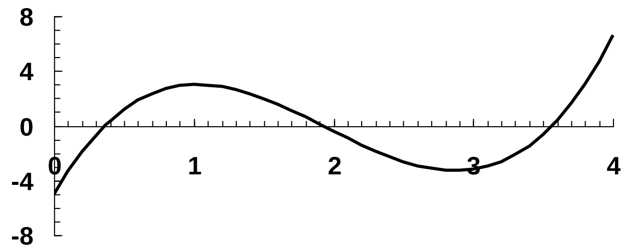
For our problem

$$g'(x) = \frac{d}{dx} 2\sin\left(\sqrt{x}\right) = \frac{1}{\sqrt{x}}\cos\left(\sqrt{x}\right)$$

The value of this quantity in the vicinity of the true root (1.9724) agrees with the ratio obtained in the table confirming that the convergence is linear and conforms to the theory.

$$g'(1.9724) = \frac{1}{\sqrt{1.9724}} \cos\left(\sqrt{1.9724}\right) = 0.1179$$

6.2 (a) Graphical



Root ≈ 3.58

(b) Fixed point

The equation can be solved in numerous ways. A simple way that converges is to solve for the x that is not raised to a power to yield

$$x = \frac{5 - 2x^3 + 11.7x^2}{17.7}$$

The resulting iterations are

i	x_i	ε_a
0	**3**	
1	**3.180791**	5.68%
2	**3.333959**	4.59%
3	**3.442543**	3.15%

(c) Newton-Raphson

i	x_i	$f(x)$	$f'(x)$	ε_a
0	**3**	-3.2	1.5	
1	**5.133333**	48.09007	55.68667	41.56%
2	**4.26975**	12.95624	27.17244	20.23%
3	**3.792934**			12.57%

(d) Secant

i	x_{i-1}	$f(x_{i-1})$	x_i	$f(x_i)$	ε_a
0	3	-3.2	**4**	6.6	
1	4	6.6	**3.326531**	-1.9688531	20.25%
2	3.326531	-1.96885	**3.481273**	-0.7959153	4.44%
3	3.481273	-0.79592	**3.586275**	0.2478695	2.93%

(e) Modified secant (δ = 0.01)

i	*x*	*f*(*x*)	*dx*	*x*+*dx*	*f*(*x*+*dx*)	*f*'(*x*)	ε_a
0	**3**	-3.2	0.03	3.03	-3.14928	1.6908	
1	**4.892595**	35.7632	0.048926	4.9415212	38.09731	47.7068	38.68%
2	**4.142949**	9.73047	0.041429	4.1843789	10.7367	24.28771	18.09%
3	**3.742316**	2.203063	0.037423	3.7797391	2.748117	14.56462	10.71%

6.3 (a) Fixed point. The equation can be solved in two ways. The way that converges is

$$x_{i+1} = \sqrt{1.8x_i + 2.5}$$

The resulting iterations are

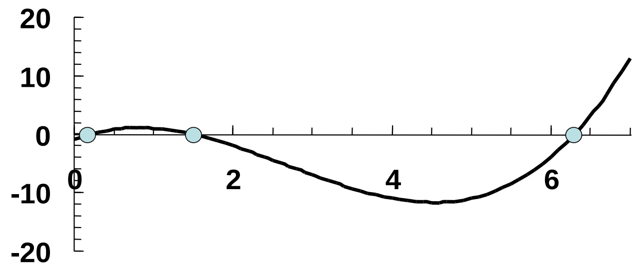
i	x_i	ε_a
0	**5**	
1	**3.391165**	47.44%
2	**2.933274**	15.61%
3	**2.789246**	5.16%
4	**2.742379**	1.71%
5	**2.726955**	0.57%
6	**2.721859**	0.19%
7	**2.720174**	0.06%
8	**2.719616**	0.02%

(b) Newton-Raphson

$$x_{i+1} = x_i - \frac{-x_i^2 + 1.8x_i + 2.5}{-2x_i + 1.8}$$

i	x_i	*f*(*x*)	*f*'(*x*)	ε_a
0	5	-13.5	-8.2	
1	3.353659	-2.71044	-4.90732	49.09%
2	2.801332	-0.30506	-3.80266	19.72%
3	2.721108	-0.00644	-3.64222	2.95%
4	2.719341	-3.1E-06	-3.63868	0.06%
5	2.719341	-7.4E-13	-3.63868	0.00%

6.4 (a) A graph of the function indicates that there are 3 real roots at approximately 0.2, 1.5 and 6.3.



(b) The Newton-Raphson method can be set up as

$$x_{i+1} = x_i - \frac{-1 + 5.5x_i - 4x_i^2 + 0.5x_i^3}{5.5 - 8x_i + 1.5x_i^2}$$

This formula can be solved iteratively to determine the three roots as summarized in the following tables:

i	x_i	*f*(*x*)	*f'*(*x*)	ε_a
0	0	-1	5.5	
1	0.181818	-0.12923	4.095041	100.000000%
2	0.213375	-0.0037	3.861294	14.789338%
3	0.214332	-3.4E-06	3.85425	0.446594%
4	0.214333	-2.8E-12	3.854244	0.000408%

i	x_i	*f*(*x*)	*f'*(*x*)	ε_a
0	2	-2	-4.5	
1	1.555556	-0.24143	-3.31481	28.571429%
2	1.482723	-0.00903	-3.06408	4.912085%
3	1.479775	-1.5E-05	-3.0536	0.199247%
4	1.479769	-4.6E-11	-3.05358	0.000342%

i	x_i	*f*(*x*)	*f'*(*x*)	ε_a
0	6	-4	11.5	
1	6.347826	0.625955	15.15974	5.479452%
2	6.306535	0.009379	14.7063	0.654728%
3	6.305898	2.22E-06	14.69934	0.010114%
4	6.305898	1.42E-13	14.69934	0.000002%

Therefore, the roots are 0.214333, 1.479769, and 6.305898.

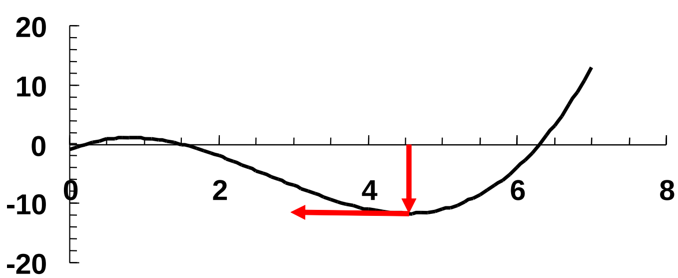
6.5 (a) The Newton-Raphson method can be set up as

$$x_{i+1} = x_i - \frac{-1 + 5.5x_i - 4x_i^2 + 0.5x_i^3}{5.5 - 8x_i + 1.5x_i^2}$$

Using an initial guess of 4.52, this formula jumps around and eventually converges on the root at 0.214333 after 21 iterations:

i	x_i	$f(x)$	$f'(x)$	ε_a
0	4.52	-11.6889	-0.0144	
1	-807.209	-2.7E+08	983842.5	100.56%
2	-537.253	-7.9E+07	437265	50.25%
3	-357.284	-2.3E+07	194341.7	50.37%
4	-237.307	-6908482	86375.8	50.56%
5	-157.325	-2046867	38390.94	50.84%
6	-104.009	-606419	17064.32	51.26%
7	-68.4715	-179640	7585.799	51.90%
8	-44.7904	-53200.9	3373.094	52.87%
9	-29.0183	-15746.4	1500.736	54.35%
10	-18.5258	-4654.8	668.5155	56.64%
11	-11.5629	-1372.39	298.5553	60.22%
12	-6.96616	-402.448	134.0203	65.99%
13	-3.96327	-116.755	60.76741	75.77%
14	-2.04193	-33.1655	28.08972	94.09%
15	-0.86123	-9.02308	13.50246	137.09%
16	-0.19298	-2.21394	7.099696	346.28%
17	0.118857	-0.40195	4.570334	262.36%
18	0.206806	-0.02922	3.909707	42.53%
19	0.21428	-0.00021	3.854637	3.49%
20	0.214333	-1E-08	3.854244	0.025%
21	0.214333	-7.3E-17	3.854244	0.000%

The reason for the behavior is depicted in the following plot. As can be seen, the guess of x = 4.52 corresponds to a near-zero negative slope of the function. Hence, the first iteration shoots to a large negative value that is far from a root.

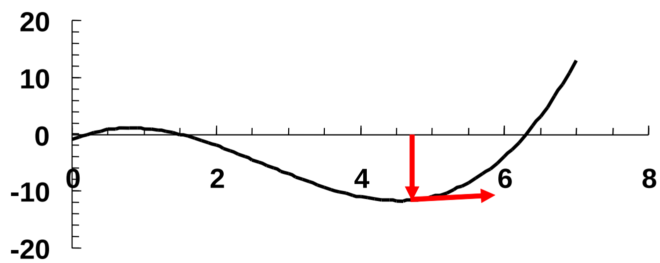


(b) Using an initial guess of 4.54, this formula jumps around and eventually converges on the root at 6.305898 after 14 iterations:

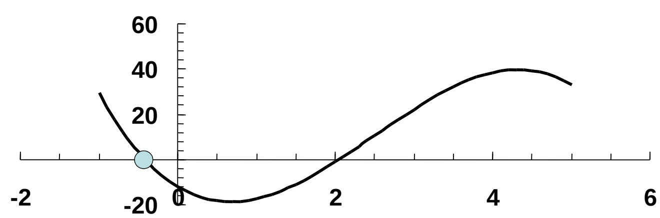
i	x_i	$f(x)$	$f'(x)$	ε_a
0	4.54	-11.6881	0.0974	
1	124.5407	904479.6	22274.75	96.35%
2	83.9351	267945.9	9901.672	48.38%
3	56.87443	79358.89	4402.556	47.58%
4	38.84879	23491.6	1958.552	46.40%
5	26.85442	6945.224	872.4043	44.66%
6	18.8934	2047.172	389.7938	42.14%

7	13.64147	598.9375	175.5027	38.50%
8	10.22877	171.854	80.61149	33.36%
9	8.096892	46.70903	39.06436	26.33%
10	6.901198	10.79053	21.73022	17.33%
11	6.40463	1.505003	15.79189	7.75%
12	6.309328	0.050492	14.73681	1.51%
13	6.305902	6.41E-05	14.69938	0.05%
14	6.305898	1.04E-10	14.69934	0.00%

The reason for the behavior is depicted in the following plot. As can be seen, the guess of x = 4.54 corresponds to a near-zero positive slope. Hence, the first iteration shoots to a large positive value that is far from a root.



6.6 (a) A graph of the function indicates that the lowest real root is approximately –0.4:



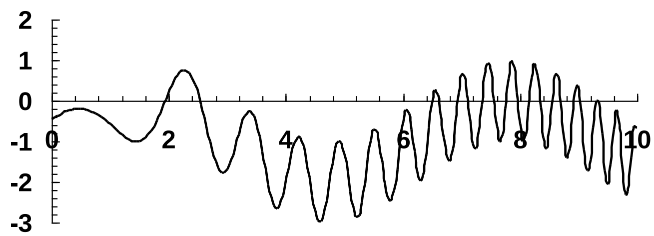
(b) The stopping criterion corresponding to 3 significant figures can be determined with Eq. 3.7 is

$$\varepsilon_s = (0.5 \times 10^{2-3})\% = 0.05\%$$

Using initial guesses of $x_{i-1} = -1$ and $x_i = -0.6$, the secant method can be iterated to this level as summarized in the following table:

i	x_{i-1}	$f(x_{i-1})$	x_i	$f(x_i)$	ε_a
0	-1	29.4	**-0.6**	7.5984	
1	-0.6	7.5984	**-0.46059**	1.72547499	30.268%
2	-0.46059	1.725475	**-0.41963**	0.15922371	9.761%
3	-0.41963	0.159224	**-0.41547**	0.00396616	1.002%
4	-0.41547	0.003966	**-0.41536**	9.539E-06	0.026%

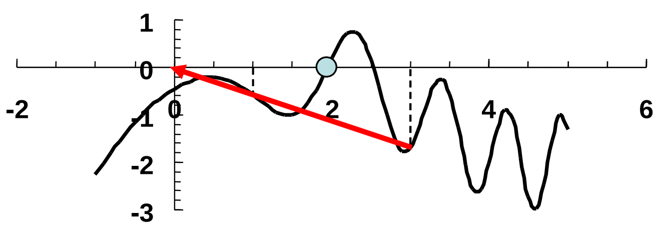
6.7 A graph of the function indicates that the first positive root occurs at about 1.9. However, the plot also indicates that there are many other positive roots.



(a) For initial guesses of x_{i-1} = 1.0 and x_i = 3.0, four iterations of the secant method yields

i	x_{i-1}	$f(x_{i-1})$	x_i	$f(x_i)$	ε_a
0	1	-0.57468	**3**	-1.697951521	
1	3	-1.69795	**-0.02321**	-0.483363437	13023.081%
2	-0.02321	-0.48336	**-1.22635**	-2.744750012	98.107%
3	-1.22635	-2.74475	**0.233951**	-0.274717273	624.189%
4	0.233951	-0.27472	**0.396366**	-0.211940326	40.976%

The result jumps to a negative value due to the poor choice of initial guesses as illustrated in the following plot:

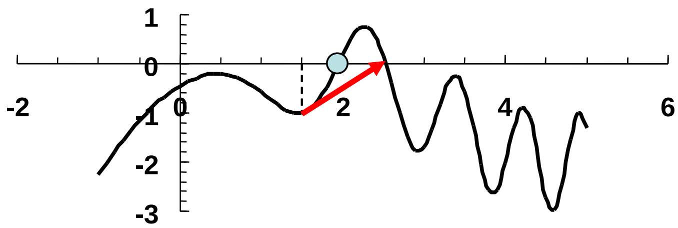


Thereafter, it seems to be converging slowly towards the lowest positive root. However, if the iterations are continued, the technique again runs into trouble when the near-zero slope at 0.5 is approached. At that point, the solution shoots far from the lowest root with the result that it eventually converges to a root at 177.26!

(b) For initial guesses of x_{i-1} = 1.0 and x_i = 3.0, four iterations of the secant method yields

i	x_{i-1}	$f(x_{i-1})$	x_i	$f(x_i)$	ε_a
0	1.5	-0.99663	**2.5**	0.1663963	
1	2.5	0.166396	**2.356929**	0.6698423	6.070%
2	2.356929	0.669842	**2.547287**	-0.0828279	7.473%
3	2.547287	-0.08283	**2.526339**	0.0314711	0.829%
4	2.526339	0.031471	**2.532107**	0.0005701	0.228%

For these guesses, the result jumps to the vicinity of the second lowest root at 2.5 as illustrated in the following plot:

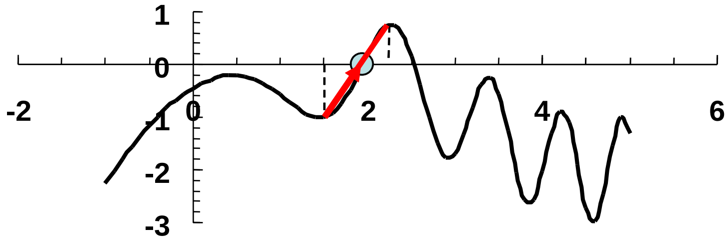


Thereafter, although the two guesses bracket the lowest root, the way that the secant method sequences the iteration results in the technique missing the lowest root.

(c) For initial guesses of x_{i-1} = 1.5 and x_i = 2.25, four iterations of the secant method yields

i	x_{i-1}	$f(x_{i-1})$	x_i	$f(x_i)$	ε_a
0	1.5	-0.996635	**2.25**	0.753821	
1	2.25	0.753821	**1.927018**	-0.061769	16.761%
2	1.927018	-0.061769	**1.951479**	0.024147	1.253%
3	1.951479	0.024147	**1.944604**	-0.000014	0.354%
4	1.944604	-0.000014	**1.944608**	0.000000	0.000%

For this case, the secant method converges rapidly on the lowest root at 1.9446 as illustrated in the following plot:



6.8 The modified secant method locates the root to the desired accuracy after one iteration:

$x_0 = 3.5$ $\qquad f(x_0) = 0.21178$

$x_0 + \delta x_0 = 3.535$ $\qquad f(x_0 + \delta x_0) = 3.054461$

$$x_1 = 3.5 - \frac{0.035(0.21178)}{3.054461 - 0.21178} = 3.497392$$

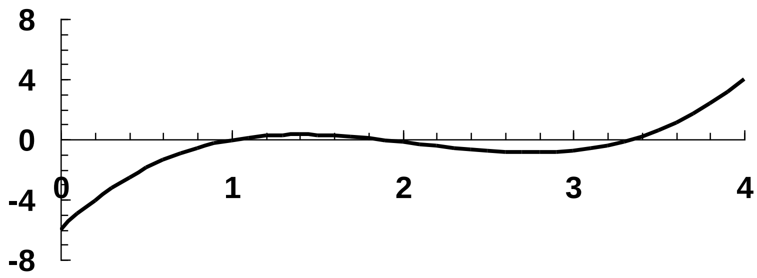
$$\varepsilon_a = \left| \frac{3.497392 - 3.5}{3.497392} \right| \times 100\% = 0.075\%$$

Note that within 3 iterations, the root is determined to 7 significant digits as summarized below:

i	x	$f(x)$	dx	$x+dx$	$f(x+dx)$	$f'(x)$	ε_a
0	**3.5**	0.21178	0.03500	3.535	3.05446	81.2194	
1	**3.497392**	0.002822	0.03497	3.532366	2.83810	81.0683	0.075%

2	**3.497358**	3.5E-05	0.03497	3.532331	2.83521	81.0662	0.001%
3	**3.497357**	4.35E-07	0.03497	3.532331	2.83518	81.0662	0.000%

6.9 (a) Graphical



Highest real root ≈ 3.3

(b) Newton-Raphson

i	x_i	$f(x)$	$f'(x)$	ε_a
0	**3.5**	0.60625	4.5125	
1	**3.365651**	0.071249	3.468997	3.992%
2	**3.345112**	0.001549	3.318537	0.614%
3	**3.344645**	7.92E-07	3.315145	0.014%

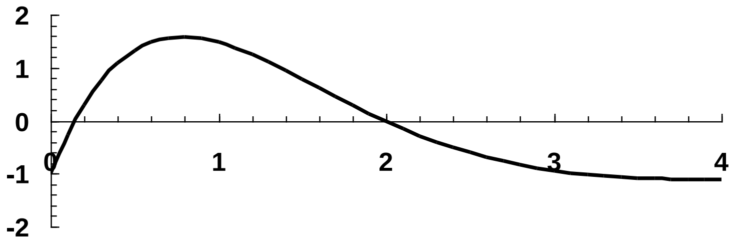
(c) Secant

i	x_{i-1}	$f(x_{i-1})$	x_i	$f(x_i)$	ε_a
0	2.5	-0.78125	**3.5**	0.60625	
1	3.5	0.60625	**3.063063**	-0.6667	14.265%
2	3.063063	-0.6667	**3.291906**	-0.16487	6.952%
3	3.291906	-0.16487	**3.367092**	0.076256	2.233%

(d) Modified secant ($\delta = 0.01$)

i	x	$x+dx$	$f(x)$	$f(x+dx)$	$f'(x)$	ε_a
0	3.5	3.535	0.60625	0.76922	4.6563	
1	3.3698	3.403498	0.085704	0.207879	3.6256	3.864%
2	3.346161	3.379623	0.005033	0.120439	3.4489	0.706%
3	3.344702	3.378149	0.000187	0.115181	3.4381	0.044%

6.10 (a) Graphical



Lowest positive real root ≈ 0.15

(b) Newton-Raphson

i	x_i	$f(x)$	$f'(x)$	ε_a
0	**0.3**	0.751414	3.910431	
1	**0.107844**	-0.22695	6.36737	178.180%
2	**0.143487**	-0.00895	5.868388	24.841%
3	**0.145012**	-1.6E-05	5.847478	1.052%

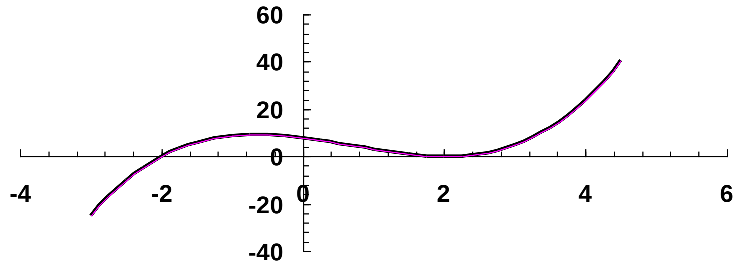
(c) Secant

i	x_{i-1}	$f(x_{i-1})$	x_i	$f(x_i)$	ε_a
0	0.5	1.32629	**0.4**	1.088279	
1	0.4	1.088279	**-0.05724**	-1.48462	798.821%
2	-0.05724	-1.48462	**0.206598**	0.334745	127.706%
3	0.206598	0.334745	**0.158055**	0.075093	30.713%
4	0.158055	0.075093	**0.144016**	-0.00585	9.748%
5	0.144016	-0.00585	**0.14503**	9E-05	0.699%

(d) Modified secant (δ = 0.01)

i	x	$x+dx$	$f(x)$	$f(x+dx)$	$f'(x)$	ε_a
0	**0.3**	0.303	0.751414	0.763094	3.893468	
1	**0.107007**	0.108077	-0.23229	-0.22547	6.371687	180.357%
2	**0.143463**	0.144897	-0.00909	-0.00069	5.858881	25.412%
3	**0.145015**	0.146465	-1.2E-06	0.008464	5.83752	1.070%
4	**0.145015**	0.146465	2.12E-09	0.008465	5.837517	0.000%
5	**0.145015**	0.146465	-3.6E-12	0.008465	5.837517	0.000%

6.11 As indicated by the following plot, a double root is located at x = 2.



(a) The standard Newton-Raphson method can be set up as

$$x_{i+1} = x_i - \frac{x_i^3 - 2x_i^2 - 4x_i + 8}{3x_i^2 - 4x_i - 4}$$

As expected, this method converges slowly as summarized in the following table:

i	x_i	*f*(*x*)	*f'*(*x*)	ε_a
0	**1.2**	2.048	-4.48	
1	**1.657143**	0.429901	-2.3902	27.586%
2	**1.837002**	0.101942	-1.22428	9.791%
3	**1.92027**	0.024921	-0.61877	4.336%
4	**1.960544**	0.006166	-0.31097	2.054%
5	**1.980371**	0.001534	-0.15588	1.001%
6	**1.99021**	0.000382	-0.07803	0.494%
7	**1.995111**	9.55E-05	-0.03904	0.246%
8	**1.997557**	2.39E-05	-0.01953	0.122%
9	**1.998779**	5.96E-06	-0.00976	0.061%
10	**1.99939**	1.49E-06	-0.00488	0.031%
11	**1.999695**	3.73E-07	-0.00244	0.015%
12	**1.999847**	9.31E-08	-0.00122	0.008%

(b) The modified Newton-Raphson method (Eq. 6.9*a*) can be set up for the double root (*m* = 2) as

$$x_{i+1} = x_i - 2\frac{x_i^3 - 2x_i^2 - 4x_i + 8}{3x_i^2 - 4x_i - 4}$$

This method converges much quicker than the standard approach in **(a)** as summarized in the following table:

i	x_i	*f*(*x*)	*f'*(*x*)	ε_a
0	**1.2**	2.048	-4.48	
1	**2.114286**	0.053738	0.953469	43.243%
2	**2.001566**	9.81E-06	0.012532	5.632%
3	**2**	3.75E-13	2.45E-06	0.078%

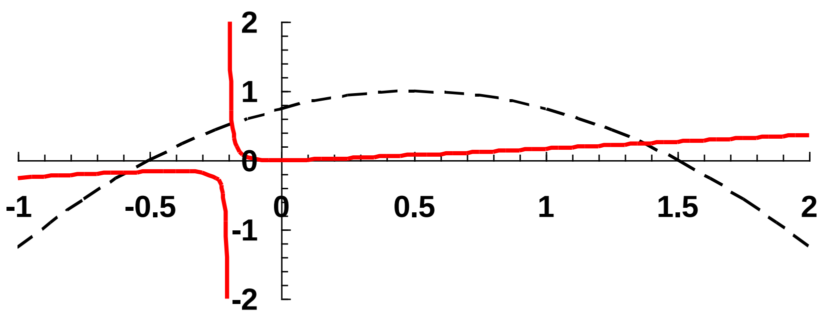
(c) The modified Newton-Raphson method (Eq. 6.13) can be set up for the double root (m = 2) as

$$x_{i+1} = x_i - \frac{(x_i^3 - 2x_i^2 - 4x_i + 8)(3x_i^2 - 4x_i - 4)}{(3x_i^2 - 4x_i - 4)^2 - (x_i^3 - 2x_i^2 - 4x_i + 8)(6x_i - 4)}$$

This method also converges much quicker than the standard approach in **(a)** as summarized in the following table:

i	x_i	$f(x)$	$f'(x)$	$f''(x)$	ε_a
0	**1.2**	2.048	-4.48	3.2	
1	**1.878788**	0.056989	-0.92562	7.272727	36.129%
2	**1.998048**	1.52E-05	-0.01561	7.988287	5.969%
3	**2**	9.09E-13	-3.8E-06	7.999997	0.098%

6.12 The functions can be plotted (y versus x). The plot indicates that there are three real roots at about (–0.6, –0.18), (–0.19, 0.6), and (1.37, 0.24).



(a) There are numerous ways to set this problem up as a fixed-point iteration. One way that converges is to solve the first equation for x and the second for y,

$$x = \sqrt{x + 0.75 - y}$$

$$y = \frac{x^2}{1 + 5x}$$

Using initial values of $x = y = 1.2$, the first iteration can be computed as:

$$x = \sqrt{1.2 + 0.75 - 1.2} = 0.866025$$

$$y = \frac{(0.866025)^2}{1 + 5(0.866025)} = 0.14071$$

Second iteration

$$x = \sqrt{0.866025 + 0.75 - 0.14071} = 1.214626$$

$$y = \frac{(1.214626)^2}{1 + 5(1.214626)} = 0.20858$$

Third iteration

$$x = \sqrt{1.214626 + 0.75 - 0.20858} = 1.325159$$
$$y = \frac{(1.325159)^2}{1 + 5(1.325159)} = 0.230277$$

Thus, the computation is converging on the root at x = 1.372065 and y = 0.239502.

Note that some other configurations are convergent and others are divergent. This exercise is intended to illustrate that although it may sometimes work, fixed-point iteration does not represent a practical general-purpose approach for solving systems of nonlinear equations.

(b) The equations to be solved are

$$u(x, y) = -x^2 + x + 0.75 - y$$
$$v(x, y) = x^2 - y - 5xy$$

The partial derivatives can be computed and evaluated at the initial guesses (x = 1.2, y = 1.2) as

$$\frac{\partial u}{\partial x} = -2x + 1 = -1.4 \qquad \frac{\partial u}{\partial y} = -1$$

$$\frac{\partial v}{\partial x} = 2x - 5y = -3.6 \qquad \frac{\partial v}{\partial y} = -1 - 5x = -7$$

The determinant of the Jacobian can be computed as

$$-1.4(-7) - (-1)(-3.6) = 6.2$$

The values of the function at the initial guesses can be computed as

$$u(1.2, 1.2) = -(1.2)^2 + 1.2 + 0.75 - 1.2 = -0.69$$
$$v(1.2, 1.2) = (1.2)^2 - 1.2 - 5(1.2)(1.2) = -6.96$$

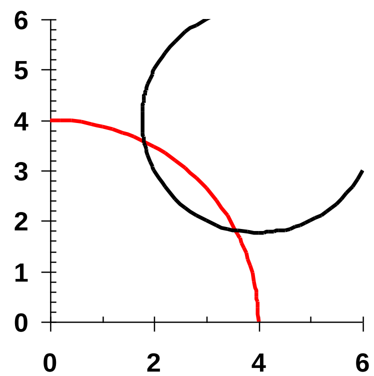
These values can be substituted into Eq. (6.21) to give

$$x = 1.2 - \frac{-0.69(-7) - (-6.96)(-1)}{6.2} = 1.543548$$
$$y = 1.2 - \frac{-6.96(-1.4) - (-0.69)(-3.6)}{6.2} = 0.0290325$$

The remaining iterations are summarized below:

i	x_i	y_i	ε_a
0	**1.2**	**1.2**	
1	**1.543548**	**0.0290325**	4033%
2	**1.394123**	**0.2228721**	86.97%
3	**1.372455**	**0.2392925**	6.86%
4	**1.372066**	**0.2395019**	0.0874%
5	**1.372065**	**0.2395019**	1.87×10^{-5}%

6.13 The functions can be plotted (y versus x). The plot indicates that there are two roots at about (1.8, 3.6) and (3.6, 1.8).



To implement the Newton-Raphson method, the equations to be solved are

$$u(x, y) = 5 - (x-4)^2 - (y-4)^2$$
$$v(x, y) = 16 - x^2 - y^2$$

The partial derivatives can be computed and evaluated at the first set of initial guesses (x = 1.8, y = 3.6) as

$$\frac{\partial u}{\partial x} = -2(x-4) = 4.4 \qquad \frac{\partial u}{\partial y} = -2(y-4) = 0.8$$

$$\frac{\partial v}{\partial x} = -2x = -3.6 \qquad \frac{\partial v}{\partial y} = -2y = -7.2$$

The determinant of the Jacobian can be computed as

$$4.4(-7.2) - 0.8(-3.6) = -28.8$$

The values of the function at the initial guesses can be computed as

$$u(1.8, 3.6) = 5 - (1.8 - 4)^2 - (3.6 - 4)^2 = 0$$
$$v(1.8, 3.6) = 16 - (1.8)^2 - (3.6)^2 = -0.2$$

These values can be substituted into Eq. (6.21) to give

$$x = 1.8 - \frac{0(-7.2) - (-0.2)(0.8)}{-28.8} = 1.805556$$
$$y = 3.6 - \frac{-0.2(4.4) - 0(-3.6)}{-28.8} = 3.569444$$

The remaining iterations are summarized below:

i	x_i	y_i	ε_a
0	**1.8**	**3.6**	
1	**1.805556**	**3.569444**	0.856%
2	**1.805829**	**3.569171**	0.0151%
3	**1.805829**	**3.569171**	2.35×10^{-6}%

For the second set of initial guesses (x = 3.6, y = 1.8), the partial derivatives can be computed and evaluated as

$$\frac{\partial u}{\partial x} = -2(x - 4) = 0.8 \qquad \frac{\partial u}{\partial y} = -2(y - 4) = 4.4$$

$$\frac{\partial v}{\partial x} = -2x = -7.2 \qquad \frac{\partial v}{\partial y} = -2y = -3.6$$

The determinant of the Jacobian can be computed as

$$0.8(-3.6) - 4.4(-7.2) = 28.8$$

The values of the function at the initial guesses can be computed as

$$u(1.8, 3.6) = 5 - (3.6 - 4)^2 - (1.8 - 4)^2 = 0$$
$$v(1.8, 3.6) = 16 - (3.6)^2 - (1.8)^2 = -0.2$$

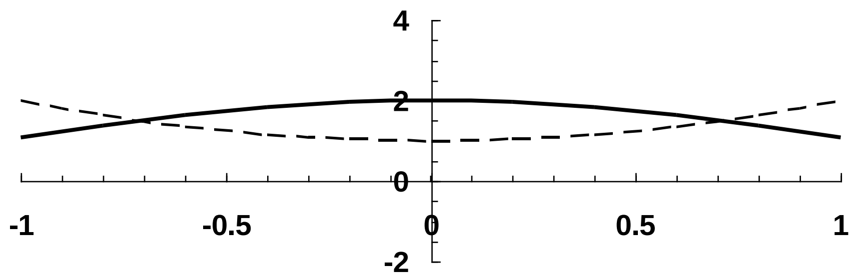
These values can be substituted into Eq. (6.21) to give

$$x = 3.6 - \frac{0(-3.6) - (-0.2)(4.4)}{28.8} = 3.569444$$
$$y = 1.8 - \frac{-0.2(0.8) - 0(-7.2)}{28.8} = 1.805556$$

The remaining iterations are summarized below:

i	x_i	y_i	ε_a
0	**3.6**	**1.8**	
1	**3.569444**	**1.805556**	0.856%
2	**3.569171**	**1.805829**	0.0151%
3	**3.569171**	**1.805829**	2.35×10^{-6}%

6.14 The functions can be plotted (y versus x). The plot indicates that there are two roots at about (–0.7, 1.5) and (0.7, 1.5).



To implement the Newton-Raphson method, the equations to be solved are

$$u(x,y) = x^2 + 1 - y$$
$$v(x,y) = 2\cos x - y$$

We will solve for the positive root. The partial derivatives can be computed and evaluated at the initial guesses (x = 0.7, y = 1.5) as

$$\frac{\partial u}{\partial x} = 2x = 1.4 \qquad \frac{\partial u}{\partial y} = -1$$

$$\frac{\partial v}{\partial x} = -2\sin(0.7) = -1.288435 \qquad \frac{\partial v}{\partial y} = -1$$

The determinant of the Jacobian can be computed as

$$1.4(-1) - (-1)(-1.288435) = -2.688435$$

The values of the function at the initial guesses can be computed as

$$u(0.7, 1.5) = (0.7)^2 + 1 - 1.5 = -0.01$$
$$v(0.7, 1.5) = 2\cos(0.7) - 1.5 = 0.0296844$$

These values can be substituted into Eq. (6.21) to give

$$x = 0.7 - \frac{-0.01(-1) - 0.0296844(-1)}{-2.688435} = 0.7147611$$
$$y = 1.5 - \frac{0.0296844(1.4) - (-0.01)(-1.288435)}{-2.688435} = 1.510666$$

The remaining iterations are summarized below:

i	x_i	y_i	ε_a
0	**0.7**	**1.5**	
1	**0.7147611**	**1.510666**	2.065%
2	**0.7146211**	**1.510683**	0.0196%
3	**0.7146211**	**1.510683**	1.76×10^{-6}%

6.15 The function to be evaluated is

$$f(c) = \frac{W}{V} - \frac{Q}{V}c - k\sqrt{c} = 1 - 0.1c - 0.25\sqrt{c}$$

Using an initial guess of $x_0 = 4$ and $\delta = 0.5$, the three iterations can be summarized as

i	*x*	*x+dx*	*f*(*x*)	*f*(*x+dx*)	*f'*(*x*)	ε_a
0	**4**	6	0.1	-0.21237	-0.15619	
1	**4.640261**	6.960392	-0.00256	-0.3556	-0.15217	13.798%
2	**4.623452**	6.935178	9.94E-05	-0.35189	-0.15226	0.364%
3	**4.624105**	6.936158	-3.8E-06	-0.35203	-0.15226	0.014%

Therefore, the root is estimated as $c = 4.624105$. This result can be checked by substituting it into the function to yield,

$$f(c) = 1 - 0.1(4.624105) - 0.25\sqrt{4.624105} = -3.8\times10^{-6}$$

6.16 Convergence can be evaluated in two ways. First, we can calculate the derivative of the right-hand side and determine whether it is greater than one. Second, we can develop a graphical representation as in Fig. 6.3.

For the first formulation, the derivative can be evaluated as

$$g'(c) = -\frac{2Q(W - Qc)}{(kV)^2} = -3.2 + 0.32c$$

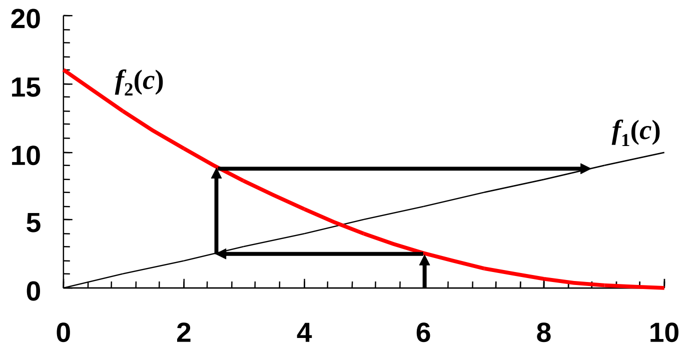
Between, $c = 2$ and 6, this ranges from –2.56 and –1.28. Therefore, because $|g'(c)| > 1$, we would expect that fixed-point iteration would be divergent.

The second way to assess divergence is to create a plot of

$$f_1(c) = c$$

$$f_2(c) = \left(\frac{W - Qc}{kV}\right)^2$$

The result also indicates divergence:



For the second formulation, the derivative can be evaluated as

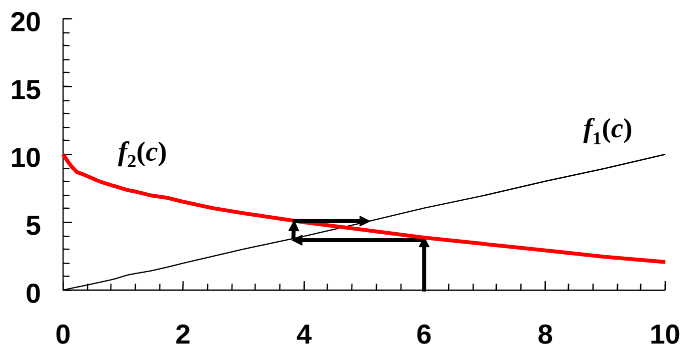
$$g'(c) = -\frac{kV}{2Q\sqrt{c}} = -\frac{1.25}{\sqrt{c}}$$

Between, c = 2 and 6, this ranges from –0.883 and –0.51. Therefore, because $|g'(c)| < 1$, we would expect that fixed-point iteration would be convergent.

The second way to assess divergence is to create a plot of

$$f_1(c) = c$$
$$f_2(c) = \frac{W - kV\sqrt{c}}{Q}$$

The result also indicates convergence:



Here are the results of using fixed-point iteration to determine the root for the second formulation.

i	x_i	ε_a	$E_{t,i}/E_{t,i-1}$
0	4		
1	5	20.0%	-0.60236
2	4.40983	13.4%	-0.56994
3	4.750101	7.2%	-0.58819
4	4.551318	4.4%	-0.57739
5	4.666545	2.5%	-0.5836
6	4.599453	1.5%	-0.57997
7	4.638416	0.8%	-0.58207
8	4.615754	0.5%	-0.58084
9	4.628923	0.3%	-0.58156
10	4.621267	0.2%	-0.58114

Notice that we have included the true error and the ratio of the true errors between iterations. The latter should be equal to $|g'(c)|$, which at the root is equal to

$$g'(4.624081) = -\frac{1.25}{\sqrt{4.624081}} = -0.5813$$

Thus, the computation verifies the theoretical result that was derived in Box 6.1 (p. 138).

6.17 Here is a VBA program to implement the Newton-Raphson algorithm and solve Example 6.3.

```
Option Explicit

Sub NewtRaph()
Dim imax As Integer, iter As Integer
Dim x0 As Double, es As Double, ea As Double
x0 = 0#
es = 0.01
imax = 20
MsgBox "Root: " & NewtR(x0, es, imax, iter, ea)
MsgBox "Iterations: " & iter
MsgBox "Estimated error: " & ea
End Sub

Function df(x)
df = -Exp(-x) - 1#
End Function

Function f(x)
f = Exp(-x) - x
End Function

Function NewtR(x0, es, imax, iter, ea)
Dim xr As Double, xrold As Double
xr = x0
iter = 0
Do
  xrold = xr
```

```
    xr = xr - f(xr) / df(xr)
    iter = iter + 1
    If (xr <> 0) Then
      ea = Abs((xr - xrold) / xr) * 100
    End If
    If ea < es Or iter >= imax Then Exit Do
  Loop
  NewtR = xr
  End Function
```

When this program is run, it yields a root of 0.5671433 after 4 iterations. The approximate error at this point is 2.21×10^{-5}%.

6.18 Here is a VBA program to implement the secant algorithm and solve Example 6.6.

```
Option Explicit

Sub SecMain()
Dim imax As Integer, iter As Integer
Dim x0 As Double, x1 As Double, xr As Double
Dim es As Double, ea As Double
x0 = 0
x1 = 1
es = 0.01
imax = 20
MsgBox "Root: " & Secant(x0, x1, xr, es, imax, iter, ea)
MsgBox "Iterations: " & iter
MsgBox "Estimated error: " & ea
End Sub

Function f(x)
f = Exp(-x) - x
End Function

Function Secant(x0, x1, xr, es, imax, iter, ea)
xr = x1
iter = 0
Do
  xr = x1 - f(x1) * (x0 - x1) / (f(x0) - f(x1))
  iter = iter + 1
  If (xr <> 0) Then
    ea = Abs((xr - x1) / xr) * 100
  End If
  If ea < es Or iter >= imax Then Exit Do
  x0 = x1
  x1 = xr
Loop
Secant = xr
End Function
```

When this program is run, it yields a root of 0.5671433 after 4 iterations. The approximate error at this point is 4.77×10^{-3} %.

For MATLAB users, here is an M-file to solve the same problem:

```
function root = secant(func,xrold,xr,es,maxit)
% secant(func,xrold,xr,es,maxit):
%   uses secant method to find the root of a function
% input:
%   func = name of function
%   xrold, xr = initial guesses
%   es = (optional) stopping criterion (%)
%   maxit = (optional) maximum allowable iterations
% output:
%   root = real root

% if necessary, assign default values
if nargin<5, maxit=50; end     %if maxit blank set to 50
if nargin<4, es=0.001; end     %if es blank set to 0.001
% Secant method
iter = 0;
while (1)
  xrn = xr - func(xr)*(xrold - xr)/(func(xrold) - func(xr));
  iter = iter + 1;
  if xrn ~= 0, ea = abs((xrn - xr)/xrn) * 100; end
  if ea <= es | iter >= maxit, break, end
  xrold = xr;
  xr = xrn;
end
root = xrn;

>> secant(inline('exp(-x)-x'),0,1)

ans =
    0.5671
```

6.19 Here is a VBA program to implement the modified secant algorithm and solve Example 6.8.

```
Option Explicit

Sub SecMod()
Dim imax As Integer, iter As Integer
Dim x As Double, es As Double, ea As Double
x = 1
es = 0.01
imax = 20
MsgBox "Root: " & ModSecant(x, es, imax, iter, ea)
MsgBox "Iterations: " & iter
MsgBox "Estimated error: " & ea
End Sub

Function f(x)
f = Exp(-x) - x
End Function

Function ModSecant(x, es, imax, iter, ea)
Dim xr As Double, xrold As Double, fr As Double
Const del As Double = 0.01
xr = x
```

```
iter = 0
Do
  xrold = xr
  fr = f(xr)
  xr = xr - fr * del * xr / (f(xr + del * xr) - fr)
  iter = iter + 1
  If (xr <> 0) Then
    ea = Abs((xr - xrold) / xr) * 100
  End If
  If ea < es Or iter >= imax Then Exit Do
Loop
ModSecant = xr
End Function
```

When this program is run, it yields a root of 0.5671433 after 4 iterations. The approximate error at this point is 2.36×10^{-5} %.

For MATLAB users, here is an M-file to solve the same problem:

```
function root = modsec(func,xr,delta,es,maxit)
% modsec(func,xr,delta,es,maxit):
%   uses the modified secant method
%   to find the root of a function
% input:
%   func = name of function
%   xr = initial guess
%   delta = perturbation fraction
%   es = (optional) stopping criterion (%)
%   maxit = (optional) maximum allowable iterations
% output:
%   root = real root

% if necessary, assign default values
if nargin<5, maxit=50; end     %if maxit blank set to 50
if nargin<4, es=0.001; end     %if es blank set to 0.001
if nargin<3, delta=1E-5; end   %if delta blank set to 0.00001
% Secant method
iter = 0;
while (1)
  xrold = xr;
  xr = xr - delta*xr*func(xr)/(func(xr+delta*xr)-func(xr));
  iter = iter + 1;
  if xr ~= 0, ea = abs((xr - xrold)/xr) * 100; end
  if ea <= es | iter >= maxit, break, end
end
root = xr;

 >> modsec(inline('exp(-x)-x'),1,.01)

 ans =
     0.5671
```

6.20 Here is a VBA program to implement the 2 equation Newton-Raphson method and solve Example 6.10.

```
Option Explicit

Sub NewtRaphSyst()
Dim imax As Integer, iter As Integer
Dim x0 As Double, y0 As Double, xr As Double
Dim yr As Double, es As Double, ea As Double
x0 = 1.5
y0 = 3.5
es = 0.01
imax = 20
Call NR2Eqs(x0, y0, xr, yr, es, imax, iter, ea)
MsgBox "x, y = " & xr & ", " & yr
MsgBox "Iterations: " & iter
MsgBox "Estimated error: " & ea
End Sub

Sub NR2Eqs(x0, y0, xr, yr, es, imax, iter, ea)
Dim J As Double, eay As Double
iter = 0
Do
  J = dudx(x0, y0) * dvdy(x0, y0) - dudy(x0, y0) * dvdx(x0, y0)
  xr = x0 - (u(x0, y0) * dvdy(x0, y0) - v(x0, y0) * dudy(x0, y0)) / J
  yr = y0 - (v(x0, y0) * dudx(x0, y0) - u(x0, y0) * dvdx(x0, y0)) / J
  iter = iter + 1
  If (xr <> 0) Then
    ea = Abs((xr - x0) / xr) * 100
  End If
  If (xr <> 0) Then
    eay = Abs((yr - y0) / yr) * 100
  End If
  If eay > ea Then ea = eay
  If ea < es Or iter >= imax Then Exit Do
  x0 = xr
  y0 = yr
Loop
End Sub

Function u(x, y)
u = x ^ 2 + x * y - 10
End Function

Function v(x, y)
v = y + 3 * x * y ^ 2 - 57
End Function

Function dudx(x, y)
dudx = 2 * x + y
End Function

Function dudy(x, y)
dudy = x
End Function

Function dvdx(x, y)
dvdx = 3 * y ^ 2
End Function

Function dvdy(x, y)
dvdy = 1 + 6 * x * y
End Function
```

Its application yields roots of $x = 2$ and $y = 3$ after 4 iterations. The approximate error at this point is 1.96×10^{-5}%.

6.21 The program from Prob. 6.20 can be set up to solve Prob. 6.11, by changing the functions to

```
Function u(x, y)
u = y + x ^ 2 - 0.75 - x
End Function

Function v(x, y)
v = x ^ 2 - 5 * x * y - y
End Function

Function dudx(x, y)
dudx = 2 * x - 1
End Function

Function dudy(x, y)
dudy = 1
End Function

Function dvdx(x, y)
dvdx = 2 * x ^ 2 - 5 * y
End Function

Function dvdy(x, y)
dvdy = -5 * x
End Function
```

Using a stopping criterion of 0.01%, the program yields $x = 1.3720655$ and $y = 0.2395017$ after 6 iterations with an approximate error of 1.89×10^{-3}%.

The program from Prob. 6.20 can be set up to solve Prob. 6.12, by changing the functions to

```
Function u(x, y)
u = (x - 4) ^ 2 + (y - 4) ^ 2 - 5
End Function

Function v(x, y)
v = x ^ 2 + y ^ 2 - 16
End Function

Function dudx(x, y)
dudx = 2 * (x - 4)
End Function

Function dudy(x, y)
dudy = 2 * (y - 4)
End Function

Function dvdx(x, y)
dvdx = 2 * x
End Function
```

```
Function dvdy(x, y)
dvdy = 2 * y
End Function
```

Using a stopping criterion of 0.01% and initial guesses of 1.8 and 3.6, the program yields x = 1.805829 and y = 3.569171 after 3 iterations with an approximate error of 2.35×10^{-6}.

Using a stopping criterion of 0.01% and initial guesses of 3.6 and 1.8, the program yields x = 3.569171 and y = 1.805829 after 3 iterations with an approximate error of 2.35×10^{-6}.

6.22 Determining the square root of a number can be formulated as a roots problem:

$$x = \sqrt{a}$$

$$x^2 = a$$

$$f(x) = x^2 - a = 0 \qquad (1)$$

The derivative of this function is

$$f'(x) = 2x \qquad (2)$$

Substituting (1) and (2) into the Newton Raphson formula (Eq. 6.6) gives

$$x = x - \frac{x^2 - a}{2x}$$

Combining terms yields the "divide and average" method,

$$x = \frac{2x(x) - x^2 + a}{2} = \frac{x^2 + a/x}{2}$$

6.23 (a) The formula for Newton-Raphson is

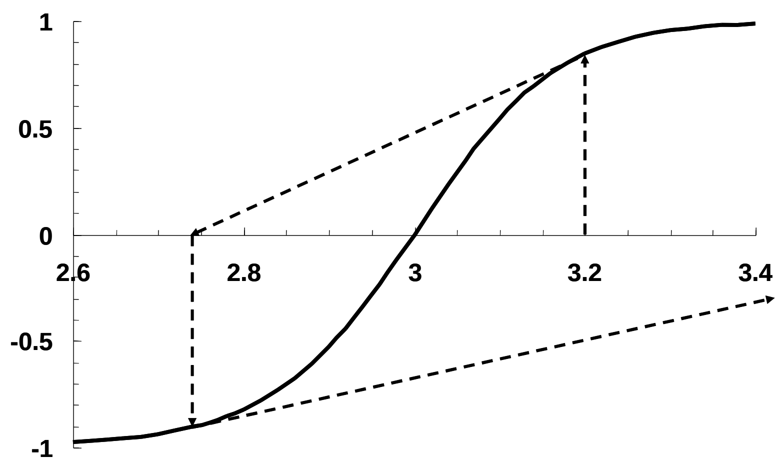
$$x_{i+1} = x_i - \frac{\tanh\left(x_i^2 - 9\right)}{2x_i \operatorname{sech}^2\left(x_i^2 - 9\right)}$$

Using an initial guess of 3.2, the iterations proceed as

iteration	x_i	$f(x_i)$	$f'(x_i)$	ε_a
0	3.2	0.845456	1.825311	
1	2.736816	–0.906910	0.971640	16.924%
2	3.670197	0.999738	0.003844	25.431%
3	–256.413			101.431%

Note that on the fourth iteration, the computation should go unstable.

(b) The solution diverges from its real root of $x = 3$. Due to the concavity of the slope, the next iteration will always diverge. The following graph illustrates how the divergence evolves.



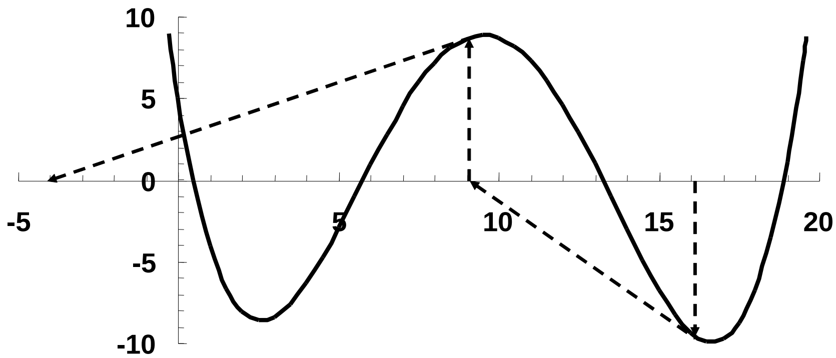
6.24 The formula for Newton-Raphson is

$$x_{i+1} = x_i - \frac{0.0074x_i^4 - 0.284x_i^3 + 3.355x_i^2 - 12.183x_i + 5}{0.0296x_i^3 - 0.852x_i^2 + 6.71x_i - 12.183}$$

Using an initial guess of 16.15, the iterations proceed as

iteration	x_i	$f(x_i)$	$f'(x_i)$	ε_a
0	16.15	–9.57445	–1.35368	
1	9.077102	8.678763	0.662596	77.920%
2	–4.02101	128.6318	–54.864	325.742%
3	–1.67645	36.24995	–25.966	139.852%
4	–0.2804	8.686147	–14.1321	497.887%
5	0.334244	1.292213	–10.0343	183.890%
6	0.463023	0.050416	–9.25584	27.813%
7	0.46847	8.81E–05	–9.22351	1.163%
8	0.46848	2.7E–10	–9.22345	0.002%

As depicted below, the iterations involve regions of the curve that have flat slopes. Hence, the solution is cast far from the roots in the vicinity of the original guess.



6.25

$f(x) = \pm\sqrt{16-(x+1)^2}+2$

$$x_{i+1} = x_i - \frac{f(x_i)(x_{i-1}-x_i)}{f(x_{i-1})-f(x_i)}$$

1st iteration

$$x_{i-1} = 0.5 \Rightarrow f(x_{i-1}) = -1.708$$

$$x_i = 3 \Rightarrow f(x_i) = 2$$

$$x_{i+1} = 3 - \frac{2(0.5-3)}{(-1.708-2)} = 1.6516$$

2nd iteration

$$x_i = 1.6516 \Rightarrow f(x_i) = -0.9948$$

$$x_{i-1} = 0.5 \Rightarrow f(x_{i-1}) = -1.46$$

$$x_{i+1} = 1.6516 - \frac{-0.9948(0.5-1.6516)}{(-1.46 - -0.9948)} = 4.1142$$

The solution diverges because the secant created by the two *x*-values yields a solution outside the function's domain.

6.26 The equation to be solved is

$$f(h) = \pi R h^2 - \left(\frac{\pi}{3}\right)h^3 - V$$

Because this equation is easy to differentiate, the Newton-Raphson is the best choice to achieve results efficiently. It can be formulated as

$$x_{i+1} = x_i - \frac{\pi R x_i^2 - \left(\frac{\pi}{3}\right) x_i^3 - V}{2\pi R x_i - \pi x_i^2}$$

or substituting the parameter values,

$$x_{i+1} = x_i - \frac{\pi(3) x_i^2 - \left(\frac{\pi}{3}\right) x_i^3 - 30}{2\pi(3) x_i - \pi x_i^2}$$

The iterations can be summarized as

iteration	x_i	$f(x_i)$	$f'(x_i)$	$\lvert\varepsilon_a\rvert$
0	3	26.54867	28.27433	
1	2.061033	0.866921	25.50452	45.558%
2	2.027042	0.003449	25.30035	1.677%
3	2.026906	5.68E-08	25.29952	0.007%

Thus, after only three iterations, the root is determined to be 2.026906 with an approximate relative error of 0.007%.