Chapter (1) Systems of Linear equation & matrices

sec 1.1 Introduction to systems of Linears equation

مقدمة في أنظمة المعادلات الخطية

Def:- A Linear equation in n variables (متغيرات) $x_1, x_2, \dots x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

where $a_1, a_2 \dots a_n, b$ are Constant [in this cource are real Constant.

Examples:-

① $2x + 3y = 7$ (Linear)

② $2x - 3y + 7z - w = 5-4$ (Linear)

③ $x_1 - x_2 + x_3 + x_4 = 0$ (Linear)

معادلة مستوى

④ $x + 3y + z = 5$ (Linear)

⑤ $3x + \sqrt{y} = 0$ (Non Linear) فيه جذر

⑥ $xz + y = 2$ (Non Linear) متغيرين مضروبين في بعض

⑦ $\sin x - y = 0$ (Non Linear) فيه $\sin x$

⑧ $x^2 + y^2 = 5$ (Non Linear) فيه تربيع

Def: A sollution of a linear equation $a_1x_1 + \dots + a_nx_n = b$ is a sequance $s_1, s_2, \dots, s_n$ in which satisfy the equation that if we subistute as $x_1 = s_1, x_2 = s_2, x_3 = s_3, \dots, x_n = s_n$ then

$$a_1s_1 + \dots + a_ns_n = b.$$

The set of sollutions of a linear equation is the set of all its sollutions.

Examples

[1] $2x = 1 \Rightarrow x = \frac{1}{2} \Rightarrow S = \{\frac{1}{2}\}$ نقطة على

[2] $4x + y = 7 \Rightarrow$ (Line)

ثبت واحد وتغيير الثاني

$x = t, \quad y = 7 - 4t$

$S = \{(t, 7-4t)\}$

[3] $x + y + z = 0 \Rightarrow (0,0), (1,-1,0), \dots$

ثبت اثنين

$x = t, \; y = s, \; z = -t - s$

$S = \{t, s, -t-s\}$ (plane)

Def: A Linear system (System of Linear equation) is a finite set of linear equations in the variable $x_1, x_2, \dots, x_n$. A sequance $s_1, \dots, s_n$ is called "A sollution of the system" if $x_1 = s_1, \dots, x_n = s_n$ is a sollution of every equation in the system.

نظام ← مجموعة من المعادلات في مجموعة من المتغيرات

← حل النظام إذا حققت كل معادلات النظام

← حل المعادلة إذا حققت كل المعادلة

Examples

(1)
$$x + y = 4$$
$$2x - y = 5$$
$$3x = 9$$
$\boxed{x = 3} \quad \boxed{y = 1}$

إذا كان معاملات المتغيرات غير متناسبة $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

$S = \{(3,1)\}$ (one solution)

(2) $2x - 3y = 2$

المعادلتين متساويتين ← $(0 = 0)$

$\frac{6x}{3} - \frac{9y}{3} = \frac{6}{3}$ القسمة على 3

$\Rightarrow 2x - 3y = 2$

$x = t, \quad y = \frac{2 - 2t}{-3}$

حلول لا نهائية

$S = \{t, \frac{2-2t}{-3}\}$ (infinite solutions)

(3)
$$x + y = 2 \quad —(1)$$
$$2x + 2y = 2 \quad —(2)$$

$$2x + 2y = 4$$
$$2x + 2y = 2$$
$$0 = 2$$

تعارض

(has no sollution)

الأنظمة الخطية:
1. لا يوجد حل
2. حل واحد
3. عدد لا محدود من الحلول

الاسئلة المطلوبة في هذا section ص ١ ← 14

(4)
$$4x_1 - 2x_2 + x_3 = 2$$
$$x_2 - x_1 + x_4 = 5$$
$$x_2 - x_3 = 6$$

تحتاج لحل المصفوفات في حل هذه المعادلات

consistant

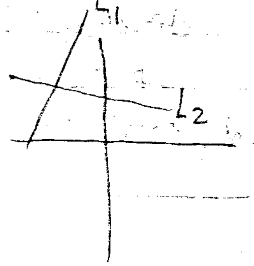
inconsistant

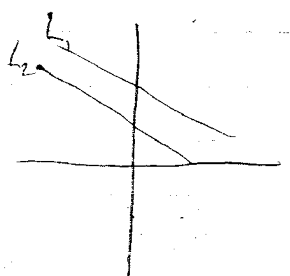
النظام الذي ليس له حل inconsistant غير متجانس

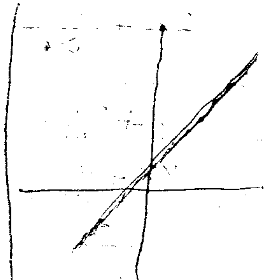
Def:- A system of equation with no sollution is called inconsistant غير متجانس , if the system
أي لا يوجد للنظام حل
has at least one sollution then it called (consistant)
متجانس

Fact:- [Will be descuss later]
every system of linear equations has
1. No sollution OR
2. exactly one sollution OR
3. infinitly many sollution.

Example:- $a_1x + b_1y = c_1$ (L_1)
$a_2x + b_2y = c_2$ (L_2)
(not both a_1, b_1 are zero)


one sollution


no sollution


infinite sollution ∞

Augmented matrix:-

Consider the system

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b_1$$
$$\vdots$$
$$a_nx_n + \cdots + a_{mn}x_{mn} = b_m$$

then the allonge

$$\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

is called the augmented matrix for the system

Examples 1)
$$x_1 - 2x_2 + 3x_3 = 0$$
$$x_2 - x_1 + x_3 = 2$$
$$x_2 - x_3 = 1$$

Sol:- الترتيب أولاً
$$x_1 - 2x_2 + 3x_3 = 0$$
$$-x_1 + x_2 + x_3 = 2$$
$$0 + x_2 - x_3 = 1$$

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

2) $x+y-z=2$
$x=y-z+3$
$x_1+y=4$

$(x\,y\,z\,x_1)$ ٤ متغيرات

$$\begin{aligned} x+y-z &= 2 \\ x-y+z &= 3 \\ y+0+x_1 &= 4 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & -1 & 0 & 2 \\ 1 & -1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 & 4 \end{bmatrix}$$

Elementary Row operation عمليات الصف البسيطة
(ERO)

ضرب الصف بعدد ليس صفر.

① Multiply a row by a non zero constant.
Multiply an equation by a nonzero constant.

تبديل صفين مكان

② Interchange two rows.
Interchange two equation

③ Add a multiple of one row to another row.
Add a multiple of an equation to another.
جمع مضاعف أحد الصفوف مع صف آخر

Example: Solve the system:-

$$\begin{aligned} x+y+2z &= 9 \\ 2x+4y-3z &= 1 \\ 3x+6y-5z &= 0 \end{aligned}$$

Sol:- $x+y+2z=9$ — ①
$2x+4y-3z=1$ — ②
$3x+6y-5z=0$ — ③

$$\begin{array}{cccc} x & y & z & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right]$$

Multiply ① by (−2)
$-2x-2y-4z=-18$
$2x+4y-3z=1$ —
$3x+6y-5z=0$

$$\begin{bmatrix} -2 & -2 & -4 & -18 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

add e_1 to e_2
$-2x-2y-4z=-18$ — ①
$0+2y-7z=-17$ — ②
$3x+6y-5z=0$ — ③

$$\begin{bmatrix} -2 & -2 & -4 & -18 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add $\left(\frac{3}{2}e_1\right)$ to e_3 →

$$\begin{bmatrix} -2 & -2 & -4 & -18 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -2 \end{bmatrix}$$

Add $\left(-\frac{3}{2}e_2\right)$ to e_3

$$\begin{bmatrix} -2 & -2 & 2 & -18 \\ 0 & 2 & -7 & -17 \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

$-\frac{1}{2}z = -\frac{3}{2} \Longrightarrow \boxed{z=3}$

$2y - 7z = -17$

$2y - 21 = -17$ $\qquad \boxed{y=2}$

$-2x - 2y - 4z = -18$

$-2x - 4 - 12 = -18$

$\boxed{x=1}$

Sol is

$\boxed{x=1}, \boxed{y=2}, \boxed{z=3}$

H.W: Solve the system in the text book

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 0 \\ x_2 - x_1 + x_3 &= 2 \\ x_2 - x_3 &= 1 \end{aligned} \Longrightarrow \begin{aligned} x_1 - 2x_2 + 3x_3 &= 0 \\ -x_1 + x_2 + x_3 &= 2 \\ x_2 - x_3 &= 1 \end{aligned}$$

Augmented matrix:-

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 1 \end{bmatrix} \Longrightarrow$$

~~$R_2 + R_1$~~ $R_2 : R_2 + R_1$

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & -1 & 4 & 2 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

$R_3 : R_3 + R_2$

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & -1 & 4 & 2 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

$3x_3 = 3$

$\boxed{x_3 = 1}$

$-x_2 + 4x_3 = 2$

$-x_2 + 4 = 2$

$-x_2 = -2$

$\boxed{x_2 = 2}$

$x_1 - 2x_2 + 3x_3 = 0$

$x_1 - 4 + 3 = 0$

$x_1 - 1 = 0$

$\boxed{x_1 = 1}$

$(x_1, x_2, x_3) = (1, 2, 1)$

اختزالات جاوس

Sec 1.2 Gaussian Elimination :-

طريقة جاوس للاختصار.

Def:-

a) A matrix is in reduced row echelone form (RREF) if it has the following properties الشروط

اذا الصف فش فيه أصفار أول عدد لازم يكون 1

1) If a row does not consist entirely of zeros then the first non zero number in the row is one we called it leading 1.

* اي سطر غير صفري اول عدد لازم يكون 1

2) If there are any rows consist of zeros then they are group together in the bottom of the matrix. السطور التي عناصرها اصفار تكون اسفل

3) In any two successive rows that don't consist of zero, then the leading one in the lower row occurs further to the right than the Leading 1 in the higher row.

$$\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix}$$

للأعمدة

4) Each column that contains a leading 1 has zeros everywhere in that column

اي عمود فيه leading 1 باقي العناصر اصفار

→ b) A matrix that has the first three propertis is said to be row echlone form. REF

Remark:-

1) Every RREF is REF

2) Any matrix has a unique RREF & many REF

Examples:- RREF

a) $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ ✓

b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

c) $\begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ ✓

d) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ V دائماً لأنها صفرية

e) $[1]$ V دائماً مصفوفة الوحدة

REF:

f) $\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ ✓

g) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

h) $\begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ ✓

NOT RREF & NOT REF

1 i) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ (4) + (3) X

j) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \leftarrow$ الرئيسية (2) + (3) X

1. every RREF is REF
2. every matrix has infinite REF but has only one RREF

k) $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ X (2) الشرط

بالضبط شكل واحد RREF

L) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ~~condit~~ X (1) X

لا يوجد leading 1

Elimination method: طريقة الاختزال

طريقة الاختزال

(1) **Guasian Elimination**: This method is using to convert a matrix to REF using the elementary raw operation. Let us do this example:

1 **Example**: Solve the system:

$$x + y + 2z = 9$$
$$2x + 4y - 3z = 1$$
$$3x + 6y - 5z = 0$$

Sol: The augmanted matrix is

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Now we convert the matrix to R.E.F.

1) Make the first entery in the firs Raw is 1. Using rearragment or devision

على مرحلة أول صف هو (1)

اجعل مرحلة أو صف هو (1) باستخدام اعادة الترتيب أو القسمة

2) Mak all number bellow (أسفل) leading 1 is zeros using rearragment (اعادة الترتيب) or E.R.O

$R_2 \longrightarrow R_2 - 2R_1$

$R_3 \longrightarrow R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

3) Repeat (كرر) step (1) (2) to the second Raw

$R_2 \longrightarrow \frac{1}{2} R_2$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & \frac{-7}{2} & \frac{-17}{2} \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

$R_3 \longrightarrow R_3 - 3R_2$

4) Do step 3) to ather rows.

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & \frac{-7}{2} & \frac{-17}{2} \\ 0 & 0 & \frac{-1}{2} & \frac{-3}{2} \end{bmatrix}$$

$R_3 \longrightarrow -2R_3$

R.E.F $\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & \frac{-7}{2} & \frac{-17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$

الحل ↓

نظام المقابل له.

The corresponding system

$$\boxed{x + y + 2z = 9}$$

$$\boxed{y - \frac{7}{2}z = -\frac{17}{2}}$$

$$\boxed{z = 3}$$

By Back subsitution.

If the augmantic matrix in R.E.F. then we can use the Back-subsitution to solve the system, as follow:-

$z = 3$

$y - \frac{7}{2}(3) = -\frac{17}{2}$

$\boxed{y = 2}$

$x + y + z = 9$

$x + 2 + 2(3) = 9$

$\boxed{x = 1}$

$\therefore$ The Sollution is $(x, y, z) = (1, 2, 3)$.

② Gauss_Gordan elimination:-

Guass & Gordan improve the last method and used this method to convert a matrix to R.R.E.F

Example $\Rightarrow$

$\begin{bmatrix} 1 & - & - \\ - & 1 & - \\ - & - & 1 \end{bmatrix}$ لو حكالك Gauss

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ لو حكالك Jordan

هنا أصل لـ REF وبعدين ابدأ بالتصفير

Example:- We will solve the last system using (G.G.E).

1) Use Gausian Elimination to convert the matrix to R.E.F

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

2) For the last leading 1 to be zeros.

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$R_2 \longrightarrow R_2 + \frac{7}{2}R_3$

$R_1 \longrightarrow R_1 + 2R_3$

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

هنا أثبت الصف الثالث واشتغل عالصف الثاني والأول.

3) Repeat 2) for the $R_1 \rightarrow R_1 - R_2$ for all below leading 1

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

④ Repeat ③ for all other leading 1.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The corresponding system:-

$x = 1$

$y = 2$

$z = 3$

#

for H.w:- Use G.E. to Solve:- then use G.G.E.

$$2x+2y-z+w=1$$
$$-x-y+2z+w+u=2$$
$$x+y-2z-u=3$$
$$z+w+u=4$$

Sol. The Augment matrix is:-

$$\begin{bmatrix} 2 & 2 & -1 & 1 & 0 & 1 \\ -1 & -1 & 2 & 1 & 1 & 2 \\ 1 & 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 1 & 1 & 4 \end{bmatrix}$$

Rearragment :- Change R_3 by R_1

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 3 \\ -1 & -1 & 2 & 1 & 1 & 2 \\ 2 & 2 & -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 4 \end{bmatrix}$$

$R_2 : R_2 + R_1$

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 & 0 & 5 \\ 2 & 2 & -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 4 \end{bmatrix}$$

الأنظمة الخطية المتجانسة (أيضاً) -

Homogenous Linear system:-

Def. A linear system of zeros Constant terms is called Homogenous linear system. it is the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ \vdots \qquad\qquad & \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

Theorem (1.2.1): A Homogenous system of linear equation with more ~~than~~ unknowns than equations has infinitly many solution.

consistant

أي نظام عددي مجاهيله أكبر من عدد معادلاته له عدد لا نهائي من الحلول

أي عكس حله -

Note:- That any H.L.S has at least one solutions $x_1 = x_2 = \cdots = x_n = 0$, which Called the trivial solution. This mean that any H.L.S has one of the following two possibles:-

1. Only the trivial solution.
2. Infinitly many solution.

Example: Solve the system:-

$$\begin{aligned} 2x_1 + 2x_2 - x_3 + x_5 &= 0 \\ -x_1 - x_2 + 2x_3 - x_5 &= 0 \\ x_3 + x_4 + x_5 &= 0 \\ x_1 + x_2 - 2x_3 - x_5 &= 0 \end{aligned}$$

The augmanted matrix:-

$$\begin{bmatrix} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + x_2 + x_5 = 0$$

$$x_3 + x_5 = 0$$

$$x_4 = 0$$

$\boxed{x_4 = 0}$

let $x_5 = t$, $x_2 = s$

$x_3 = -t$, $x_1 = -s - t$

Solutions is:-

$$(x_1, x_2, x_3, x_4, x_5) = (-s-t, s, -t, 0, t)$$

Remark: The Albitary values, s, t, r --- etc are called parameter.
عشوائية

Section 1.3

ك ← ص ← ط ← نسبية ← ح

$C \leftarrow R \leftarrow Q \leftarrow Z \leftarrow N$.

المصفوفات
العمليات عليها

Matrices & Matrix operation

المصفوفة شكل مستطيل شكل من الأعداد

def: A matrix is A rectangular array of numbers called entries.
مدخلات

Examples: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $[2 \ 1 \ 0 \ 4 \ 5]$, $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $[5]$

الخطوط الأفقية

Remark (1) The horizontal lines in the matrix are called Rows (صفوف) and vertical lines are called Columns (عمود).

(2) The size of a matrix is described (يوصف) by its rows & columns as Numbers of rows X a number of colums

العمود X الصف

\# Rows X \# Column

likes:

3×3, 2×3, 1×5, 2×1

(3) We used capital letters $(A, B, C, \ldots)$ to denote the matrices.

(4) The entry in the row i and column j of a matrix A will be denoted by $a_{(ij)}$ ← عمود / صف

That if (A) is $m\times n$ matric, then we can write

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix}$$

main diagonal.

and we can write $A = [a_{ij}]_{m\times n}$ OR $[a_{ij}]$ also, we can write $a_{ij} = (A)_{ij}$

def. A matrix A with n rows and n columns is called a square matrix of order n, and $a_{11}, a_{22}, a_{33}, a_{44}, \ldots a_{nn}$ are said to be in the main diagonal.
القطر الرئيسي

Ex:-

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 4 \\ 5 & -1 & 0 \\ 2 & 3 & 7 \end{bmatrix}, \quad C = [1]$$

$a_{22} = 4$, $b_{31} = 2$,

$b_{13} = 4$ صف ثم عمود

Operations of matrics:-

المساواة

1) Equality :-

def:- Two matrices A & B are called be equal if they have the same size and their corresponding entries one equal

That if $A = [a_{ij}]$, $B = [b_{ij}]$ then $A = B$

iff $a_{ij} = b_{ij}$

Examples:-

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \end{bmatrix}$$

$A = B$

$A \neq C$

$A \neq D$

انتبه لازم نفس الحجم

$A = B$

2) Sum and difference:-

def:- If A & B are matrices of the same size, the sum $A+B$ (difference $A-B$) is the matrix obtianed by adding (substracting) the corresponding entries of A and B.

We cannot do this for matrices with differents size in matrix notation

$$A \pm B = [a_{ij} \pm b_{ij}]_{mn}$$

Example:-

Example. $A = \begin{bmatrix} 1 & 2 & 4 \\ 5 & 6 & -1 \end{bmatrix}$

$B = \begin{bmatrix} 2 & 3 & -1 \\ 2 & +4 & 5 \end{bmatrix}$

$C = \begin{bmatrix} -1 & 0 \\ 2 & \sqrt{3} \end{bmatrix}$

$A+B = \begin{bmatrix} 1+2 & 2+3 & 4-1 \\ 5+2 & 6+4 & 5-1 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 3 \\ 7 & 10 & 4 \end{bmatrix}$

$A-B = \begin{bmatrix} 1-2 & 2-3 & 4+1 \\ 5-2 & 6-4 & -5-1 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 5 \\ 3 & 2 & -6 \end{bmatrix}$

$A+C =$ NOT (X)

undefined

ليس من نفس الحجم

③ Scalar Product :-

If A is any matrix and c is any scalar ($c \in R$) then the product cA is the matrix obtioned by multiplying each entry of A by c.

cA is called the Scallar multiple of A. In matrix notation.

$cA = c\,[a_{ij}]_{m\times n} = [c a_{ij}]_{m\times n}$

Ex:- If $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & -1 & 0 \end{bmatrix}$ the

$\frac{1}{2}A = \begin{bmatrix} 1 & \frac{3}{2} & 2 \\ \frac{1}{2} & \frac{-1}{2} & 0 \end{bmatrix}$

$-2A = \begin{bmatrix} -4 & -6 & -8 \\ -2 & 2 & 0 \end{bmatrix}$

$0A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

ضرب المصفوفات

(4) Produt of matrices:-

If A is a matrix with size $m \times r$ and B is a matrix with size $r \times n$ then the product AB is the $m \times n$ matrix whose entries are determined as follows:-

To find the entry in the row i and column j of AB we single the row i of A and the column k of B and multiplying to corresponding the entries from the row and column together and add up the resulting product. That is If

$$A = [a_{ij}]_{m \times r}, \quad B = [b_{ik}]_{r \times n}, \text{ then}$$

$$AB = [c_{ik}]_{m \times n}, \text{ where}$$

$$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots + a_{ir}b_{rk}$$

Example:

$$A = \underset{2\times 3}{\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}}, \quad B = \underset{3\times 4}{\begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}}$$

$$C = \begin{bmatrix} 2 & 1 \\ 5 & 0 \\ -1 & 0 \end{bmatrix} \quad 3\times 2$$

Sol.- $A_{2\times 3}$, $B_{3\times 4}$, $C = 3\times 2$

Sol

$AB = (\checkmark)$ يصلح

$BA = (\times)$ لا يصلح

$AC = (\checkmark)$ يصلح

$CA = (\checkmark)$ يصلح

$BC = (\times)$ لا يصلح

$CB = (\times)$ لا يصلح

نحضر مثال - عملية الضرب على المصفوفات التالية

$$AB = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \times \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{12} & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

$2\times 3 \to 3\times 2 \to 2\times 2$ (2×4)

$$AC = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ 5 & 0 \\ -1 & 0 \end{bmatrix}$$

$$AC = \begin{bmatrix} 8 & 1 \\ 34 & 2 \end{bmatrix} \quad 2\times 2$$

$$CA = \begin{bmatrix} 2 & 1 \\ 5 & 0 \\ -1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}$$

<u>Remark</u>:- We can find a fixed row or a fixed column of AB as:-

j_{th} Column of $AB = A\,[j_{th} \text{ Column of } B]$

i_{th} row of $AB = [i_{th} \text{ of } A]\, B$

كتابة النظام الخطي باستخدام المصفوفات

Matrix form of a Linear system:-

Consider the system:-

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

We can write the previous system as:-

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

مصفوفة المعاملات

$$A_{m\times 1} \quad X \quad = b_{m\times 1}$$

We called A the Coeffisient matrix. Clearly the augmented matrix of this system is:-

$$\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_n \end{array}\right] = [A \mid b]$$

المدور

Transpose, Trace:- مجموع عناصر القطر الرئيسي

1 def:- If A is any matrix $(m\times n)$, then the <u>transpos</u> of A denoted by A^T is the $(n\times m)$ matrix results by interchange the rows and the columns of A. in matrix notation

$$A^T = [a_{ji}]_{n\times m}$$ شرط ما بصورة مربعة

مصفوفة مربعة

2 If A is a <u>square matrix</u> then the <u>trace</u> of A denoted by $tr(A)$ is the sum of the entries in the main diagonal.

<u>Example</u>:- 1 find A^T if

مهم

(a) $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(c) $A = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}$

(d) $A = [1] \Rightarrow A^T = [1]$

2) Find tr(A)

(a) $A = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix} \Rightarrow tr(A) = 11$

(b) $A = [2] \Rightarrow tr(A) = 2$

(c) $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ 4 & 0 \end{bmatrix} \Rightarrow tr(A) = \times$

No square matrix

كلمة سر ابن

خطه ابراهيم

ا سناء

1) a – c جمع فقط

2) d – f ضرب مع جمع مباح

3) h → جمع x بس

Section 1.4.1 Inverses, Rules of matrix Arithmetic:-

Properties of matrix operation:- خصائص العمليات على المصفوفات

(illegible Arabic line)

Theorem 1.4.1: Assuming that the size of the matrices are such that the indicated operations can be performed, then the following rules are valid:-

الجمع عملية تبديلية

a) (i) $A+B = B+A$ (commutitive of addition)

الضرب عملية تجميعية

b) (ii) $A+(B+C) = (A+B)+C$ (assiciativ of addition)

c (iii) $A(BC) = (AB)C$ × (assiciative of multiplication) تجميعية الضرب ليست

d (iv) $A(B+C) = AB+AC$ (left distribution law)

e (v) $(B+C)A = BA+CA$ (right distribution law)

f (vi) $(B-C)A = BA - CA$ × ليست اساسية ليست

g (vii) $a(B-C) = aB - aC$ ليست تحصيلية

h (viii) $(a+b)A = aA + bA$ ناتج

k (ix) $(a-b)C = aC - bC$

r (x) $a(bC) = (ab)(C)$

(xi) $a(BC) = (aB)(C) = B(aC)$

Proof:- مطلوب

(a , (d , (h

(a) $A+B = B+A$

let $A = [a_{ij}]_{m\times n}$

$B = [b_{ij}]_{m\times n}$

$A+B = [a_{ij}]_{m\times n} + [b_{ij}]_{m\times n}$

$= [a_{ij} + b_{ij}]_{m\times n}$

$= [b_{ij} + a_{ij}]_{m\times n}$

$= [b_{ij}]_{m\times n} + [a_{ij}]_{m\times n}$

$\boxed{A+B = B+A}$ ※

(e) $(B+C)A = BA + CA$

We know that the size ① determine such that the operation are performed so, assume B and C are $m\times r$ matrix, A is $r\times n$ matrix.

$[(B+C)A]_{ik} = (b_{i1}+c_{i1})a_{1k} + (b_{i2}+c_{i2})a_{2k} + \cdots + (b_{ir}+c_{ir})a_{rk}$

$= (b_{i1}a_{1k} + a_{i2}a_{2k} + \cdots + b_{ir}a_{rk}) + (c_{i1}a_{1k} + c_{i2}a_{2k} + c_{ir}a_{rk})$

$= [BA]_{ik} + [CA]_{ik}$

$= [BA + CA]_{ik}$

$\Rightarrow (B+C)A = BA + CA$ ※

Ex: Given that $A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$

$AB = \begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix}$

$BA = \begin{bmatrix} 3 & 6 \\ -5 & 0 \end{bmatrix}$

$AB \neq BA$

ليس ضروري

Remark:- It is not nessecary that $\forall A, B$

مش ضروري
أن يتساووا

$AB = BA$

def:- A matrix which all of it entries zero's is called a zero matrix

المصفوفة التي مدخلاتها أصفار : المصفوفة الصفرية

Ex.

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $[0]$, $[0\ 0\ 0]$

Example:- Given that

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix}$$

then,

$$AB = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$$

$$AC = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$$

$$AD = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

اللوضع أن التوسيعة ع الصفوفات

~~Bm~~ Notice

$B \neq C$ but $AB = AC$

$A \neq \overset{\text{zero matrix}}{0} \neq D$ but ~~By~~ $AD = 0$

$\therefore AB = AC \nRightarrow B = C$

مش شرط دايما

Remark:-

1) The Cancelation law doesn't hold in matrices multiplication, that If $AB = AC$, then it is not necessary that $B = C$

2) If $AB = 0$, then it is not necessary that $A = 0$ or $B = 0$

Theorem 1.4.2

Assuming that the size is determined such that the indicated operation can be performed, the following rules of matrix are valid:-

a) $A + 0 = 0 + A = A$

b) $A - A = 0$

c) $0 - A = -A$

d) $A\overset{\text{صفرية}}{0} = \overset{\text{صفرية}}{0}A = \overset{\text{صفرية}}{0}$

$\overset{\text{scalar}}{0}A =$

Proof:-

d)

$$AO = A(0) = A(0+0)$$

$$0 + AO = AO + AO$$

$$0 + \cancel{AO} - \cancel{AO} = AO + \cancel{AO} - \cancel{AO}$$

بنفس الطريقة $\quad 0 = AO$

(So, as $0 = 0A$)

Proof (d) $\quad 0A = 0$

$$0[a_{ij}]_{m\times n} = [0a_{ij}]_{m\times n}$$

$$[0]_{m\times n} = 0$$

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

def:- A sequare matrix with one's on the main diagonal and zeros otherwise is called the Identity matrix (مصفوفة الوحدة) and denoted by I_n if its size is $n\times n$. لازم تكون مربعة

We can see that if A is $m\times n$ matrix then

والمصفوفة الوحدة هي المحايد الضربي للمصفوفات

$$I_m A = A$$

$$A I_n = A$$

عناصر القطر الرئيسي عبارة عن 1

// // الباقي عبارة عن 0

مصفوفة الوحدة

Example:- $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $[1]$

Theorem: 1.4.3 If R is the R.REf of the $m\times n$ matrix A then, R has a row of zeros or it is the identity matrix I_n

$$A_{n\times n} \xrightarrow{RREf} R \to \begin{cases} \text{row of zeros } \text{صف من الأصفار} \\ I_n \leftarrow \text{مصفوفة الوحدة} \end{cases}$$

أي مصفوفة مربعة في حالة RREf تكون على حالة من حالتين

1) اما أن تكون على صف من الأصفار

2) أو تكون عبارة عن مصفوفة الوحدة

Proof:-

If R hasn't a row of zeros, Then all Rows has leading one So we have n leading one & we put these leading one on the main diagonal (القطر الرئيسي) and every colum has leading one and so other entries are zero's on this column.

So we have I_n.

def:-

If A is a square matrix (مصفوفة مربعة) and there is a matrix B such that $\underline{AB = BA = I}$, then A is said to be invirtable (قابل للعكس) and B is called an inverse of A. We will write $B = A^{-1}$. If no such B found then A is said to be singular (منفردة).

$$\begin{array}{ccc} & A & \\ \swarrow & & \searrow \\ \text{invertable} & & \text{singular} \end{array}$$

قابل للعكس — منفردة

غير قابلة للانعكاس

Example:- [1] $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \Big] \; AB = BA = I$$

A is invertable and $A^{-1} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$

[2] $A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$ is A invertable:-

Sol:

assume $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$

then the column on ~~BA~~ $BA = B$ (the third column on A)

$$= \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So, $BA = \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ c_{31} & c_{32} & 0 \end{bmatrix}$

So, $BA \neq I$, $\forall B$, 3×3 matrix

So A is singular.

Properties of Inverse:-

Theorem 14.4

If B and C are both inverses of A then $B = C$ (Inverse is uniqe) وحيد المعكوس

Proof:-

$B = BI = B(AC)$ ($AC = I$ since C is inverse to A)

$\quad = (BA)C$

$B = IC = C$ ($BA = I$ since B is inverse of A)

$\boxed{\therefore B = C}$

Theorem:- 1.4.5

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad - bc \neq 0$, then

A is invertable and $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Proof:-

Excersise (show that $AA^{-1} = I = A^{-1}A$)

$$AA^{-1} = I = A^{-1}A$$

اعكس القطر الرئيسي

وبدل اشارات القطر الفرعي

$AI = A$

Theorem 1.4.6

If A and B are invertable matrices of same (بنفس) size then AB is invertable and

$(AB)^{-1} = B^{-1}A^{-1}$ (التوزيع بشكل عكسي)

$$(AB)\cdot(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}BI = B^{-1}B = I$$

$$\text{So, } (AB)^{-1} = B^{-1}A^{-1}$$

Example:- $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$, $B^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix}$

$AB = \begin{bmatrix} 7 & 6 \\ 9 & 8 \end{bmatrix}$

$(7\times 8 - 9\times 6) = 2$

$(AB)^{-1} = \frac{1}{2}\begin{bmatrix} 8 & -6 \\ -9 & 7 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}$

$B^{-1}A^{-1} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}$

$A^{-1}B^{-1} = \begin{bmatrix} 5 & 0 \\ -2 & \frac{5}{2} \end{bmatrix}$

$$\boxed{(AB)^{-1} = B^{-1}A^{-1}} \quad \#$$

Def:- If A is a squar matrin the we defind the non negative power of A as:-

$$A^0 = I \quad , \quad A^n = \underbrace{A.A.A.\ldots.A}_{n\text{-times}}$$

If A is invertible, then the negative powels to be

$$A^{-n} = (A^{-1})^n = \underbrace{A^{-1}.A^{-1}.A^{-1}.\ldots.A^{-1}}_{n\text{ times}}$$

nonnegative $= (0, 1, \ldots)$

Theorem 1.4.7

If A is a squance matrin and r and s are integers then:

non negative

$$A^r \cdot A^s = A^{r+s} \quad , \quad (A^r)^s = A^{rs}$$

Proof:-

$$A^r A^s = (\underbrace{A.A.\ldots A}_{r\text{-times}}).(\underbrace{A.A.A.\ldots A}_{s\text{-times}})$$

$$= (\underbrace{A.A.A.\ldots}_{(s+r)\text{-times}}) = A^{r+s}$$

$$(A^r)^s = (\underbrace{A^r.A^r.A^r.\ldots A^r}_{s\text{-times}}) = (\overbrace{A.A.\ldots A}^{r\text{-time}})(\overbrace{A.A.A.\ldots A}^{r\text{-times}}) \ldots (\underbrace{A.A.\ldots A}_{r\text{-times}})$$

$$(\overbrace{A.A.\ldots A}^{rs\text{-times}}) = A^{rs}$$

Theorem 1.4.8

If A is invertable matrin then

a) A^{-1} is invertable and $(A^{-1})^{-1} = A$

b) A^n is invertable and $(A^n)^{-1} = (A^{-1})^n = A^{-n}$

c) for non zero scalar k, kA is invertable and $(kA)^{-1} = \frac{1}{k} A^{-1}$

Proof:- a) since $A^{-1}A = AA^{-1} = I \Longrightarrow (A^{-1})^{-1} = A$

b) $$(A^n)(A^{-1})^n = (\underbrace{A.A.\ldots A}_{n\text{-times}})(\underbrace{A^{-1}.A^{-1}.A^{-1}.\ldots A^{-1}}_{n\text{-times}})$$

$$(A.A^{-1})(AA^{-1})(A^{-1}A) = I$$

$$(\underbrace{A.A.A.\ldots A}_{n-1})(\underbrace{A^{-1}A^{-1}\ldots A}_{n-1}) =$$

نشيلة الإثنين = أينما بجانب

$$AA^{-1} = I$$

~~So as~~ $=$

So as $(A^{n})^{-1} = (A^{-1})^{n}$

(c) $(k\,A)\left(\frac{1}{k}\,A^{-1}\right) = k\cdot\frac{1}{k}\,(A\cdot A^{-1}) = 1\cdot I = I$

$\left(\frac{1}{k}\,A^{-1}\right)(kA) = \frac{1}{k}\cdot k\cdot A^{-1}\cdot A = 1\cdot I = I$

$(kA)^{-1} = \frac{1}{k}\cdot A^{-1}$

Example:-

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{1}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

$$A^{3} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\cdot\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & [illegible] \end{bmatrix}$$

def:- If A is مربعة $m\times m$ matrix, and $P(x) = a_0 + a_1x + \dots a_n x^n$ is any Polynomial then we can define

$$P(A) = a_0 I + a_1 A + a_2 A^2 + \dots + a_n A^n$$

Clearly, $P(A)$ is $m\times m$ matrix.

Example:- $A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$, $P(x) = 2x^2 - 3x + 4$

$$P(A) = 2A^2 - 3A + 4I$$

$$= 2\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} - 3\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 9 & 2 \\ 0 & 13 \end{bmatrix}}$$

الفكرة انك تتعب تعوض بدل
الرقم عشوائيات

Properties of transpose:-

Theorem 1.4.9

If the size of matrices are such that the stated operations can be performed, then :-

a) $((A^T))^T = A$, b) $(A \pm B)^T = A^T \pm B^T$

c) $(kA)^T = k\,A^T$, d) $(AB)^T = B^T A^T$.

غير مطلوب الاثبات

Theorem 1.4.10 If A is invertable, then A^T is also invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

Proof:-

$$(A^T)(A^T)^{-1} = (A^{-1}A)^T = (I)^T = I$$

$$(A^{-1})^T(A^T) = (AA^{-1})^T = (I)^T = I$$

So, $(A^T)^{-1} = (A^{-1})^T$

Example:- If $A = \begin{bmatrix} -5 & -3 \\ 2 & 1 \end{bmatrix}$

$$A^T = \begin{bmatrix} -5 & 2 \\ -3 & 1 \end{bmatrix} \longrightarrow (A^T)^{-1} = \begin{bmatrix} 1 & -2 \\ 3 & -5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{1}\begin{bmatrix} 1 & 3 \\ -2 & -5 \end{bmatrix} \rightarrow (A^{-1})^T \begin{bmatrix} 1 & -2 \\ 3 & -5 \end{bmatrix}$$

$\boxed{\text{The End of sec 1.4}}$

إجراء عملية واحدة على مصفوفة الوحدة

sec 1.5 Elementary (المصفوفات البسيطة) Matrices and a method (طريقة) for finding (لإيجاد) $\boxed{A^{-1}}$:-

مصفوفة الأولية هي مصفوفة مربعة يمكن الحصول عليها بإجراء عملية صف بسيطة واحدة على مصفوفة الوحدة

def:-

An $n\times n$ matrix is called elementary matrix if it can (إيجادها) obtained **from** $n\times n$ **Identity** matrix I_n by a single Elementary Raw Operation.

Example:- $A = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$ Elementary matrix (✓) because $I_n \xrightarrow{R_2 \to -3R_2} E$

B) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ (✓) $I_n \xrightarrow{R_2 \Leftrightarrow R_4} E$

c) $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (✓). $I_n \xrightarrow{R_1 \to 3R_3 + R} E$

d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓ $I_n \xrightarrow{R_1 \to 1R_1} E$

e) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ X (2 operation)

d) $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ X (2 operation)

عمليتين

the ...
we supo...
we all

Theorem 1.5.1

If the elementary matrix E results from forming a certain operation on I_m and If A is $m \times n$ matrix, then the product EA is the matrix result when this same row operation ~~operation~~ Performed on A.

$$\boxed{I_m} \xrightarrow{ERO} \boxed{E}$$

$$A \xrightarrow{ERO} EA \xrightarrow{O_2} E_1 EA$$

Example:- let $A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}$, $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ I_3

$$EA = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{bmatrix}$$

$I \xrightarrow{(R_3 \to R_3+3R_1)} E$

$$A \xrightarrow[E]{R_3 \to R_3+3R_1} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{bmatrix}$$

Inverse Row operation:-

Every elementary operation that product the elementary matrix E has an Inverse operation obtain the inverse of E

$$E^{-1} \xleftarrow[\text{عكس العملية}]{O} I \xrightarrow[\text{عملية}]{O} E$$

العملية Row operation Product E	عكس العملية R.O. Product E^{-1}
1) multiply a row i by $c \neq 0$ ضرب	① Multiply by $\frac{1}{c}$
2) Interchange row i and j تبديل	② Interchang i and j بين الشيئين
3) Add c times row i to row j اضافة	③ Add $-c$ times of i to j. ازالة

Examples:- $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to 3R_2} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

$R_2 \to \frac{1}{3} R_2 \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 \Leftrightarrow R_2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \xrightarrow{R_1 \Leftrightarrow R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1+2R_2} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1-2R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Section 1.5

Theorem 1.5.2

أي مصفوفة أولية [illegible]

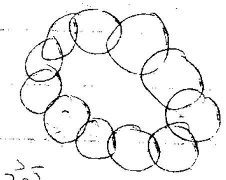
Every elementary matrix is invertable and the invers is also an Elementary matrix.

$I_n \xrightarrow{ER} E$

$I_n \xrightarrow{ER^{-1}} E^{-1}$

Theorem: 1.5.3 نظرية رئيسية

If A is an $n\times n$ matrix, then the following statment are equivellent (T.F.S.E) متكافئة



$$\begin{tikzcd} a \arrow[r, bend left, "\text{تؤدي}"] & b \arrow[r, bend left, "\text{تؤدي}"] & c \arrow[r, bend left, "\text{تؤدي}"] & d \arrow[lll, bend left] \end{tikzcd}$$

ركز على الإثباتات

a) A is invertable.

b) $Ax=0$ has only the ~~interiat~~ trivial solution.

c) The reduced row echlone form of A is I_n

d) A is expressible as a product of Elementary matrics.

Proof:-

a) $\longrightarrow$ b)

assume A is invertable so A^{-1} exist.

now, If $Ax=0$

$$A^{-1}Ax = A^{-1}0$$

$$\boxed{x=0}$$

So we have only the trivial solution.

b) $\longrightarrow$ c)

assume $Ax=0$ has only the trivial solution then the system $a_{11}x_1+a_{12}x_2+\dots\dots+a_{1n}x_n=0$

$\vdots$

$a_{n1}x_n+a_{n2}x_2+\dots a_{nn}x_n=0$

After solution $\longrightarrow$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & 0 \\ | & | & | & & | & 0 \\ | & | & | & & | & 0 \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{n} & 0 \end{bmatrix} \xrightarrow[\text{R.R.E.F}]{\text{G.G.E}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$x_1, x_2, x_3, \dots, x_n = 0$

$A \xrightarrow{\text{G.G.E}} I$

So, the RREF of A is I_n.

c) $\longrightarrow$ d) assume that RREF of A is I_n

نحصل

So, we can obtain I_n from A by Perform a finite sequance of E.R.O and

since every E.R.O on A is done as product elementary matrix ~~E~~ of A is done as product of elementary matrix (~~E of A~~) $\mathcal{E}$ of A.

So we can set

$$E_m E_{m-1} E_{m-2} \cdots E_3 E_2 E_1 A = I_n$$

$$A = E_1^{-1} E_2^{-1} \cdots E_{m-2}^{-1} E_{m-1}^{-1} E_m^{-1}$$

$$A = \grave{E}_1 \grave{E}_2 \cdots \grave{E}_{m-2} \grave{E}_{m-1} \grave{E}_m$$

So A ~~as operati~~ is a product if elementary matrices.

d) $\longrightarrow$ a)

assume A is a product of elementary matrices

So, $\boxed{A} = E_1 E_2 \cdots E_m$

but every E_i is inverdtble

So A is invertable.

Def:- Two matrices A and B are said to be row equivelent if we can obtain B from A using a finite number of elementary row operation.

أهم التعريف

$$A \xrightarrow{E.R.O} B$$

$$B = E_1 E_2 \cdots E_m A$$

طريقة لعكس المصفوفة

A method for inversing matrices:-

from theorem 1.5.3, A is invertable If

$$A = E_1E_2 \cdots E_{m-1}E_{m-2}E_m \text{, where}$$

E_i is elementary matrices

$$A^{-1} = (E_1E_2 \cdots E_m)^{-1} = E_m^{-1}E_{m-1}^{-1} \cdots E_2^{-1}E_1^{-1}$$

RRF

So, we will begin with

$$[A|I_n] \xrightarrow{E.R.O} [I_n|B] \text{ or } A^{-1}$$

B is A^{-1}

Example:- Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

دايماً بدك تشتغل هنا

لازم تشتغل هنا

باستخدام طريقة ...

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 + 2R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

$R_3 \to -R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 + 3R_3]{R_1 \to R_1 - 3R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$R_1 \to R_1 - 2R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

So

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

Example: Show that $A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$ is sigular.

Sol:

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & 9 & -2 & 1 & 0 \\ 0 & 8 & 4 & 1 & 0 & 1 \end{array}\right]$$

$$R_3 \to R_3 + R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

We can not obtain I_3 in the first part

So A is a singular

not inversible

Ex 5/7 page 22

Section 1.6 Further results on systems of equation and invertablity:-

القابلية للعكس

Theorem:- 1.6.1

Every system of linear equations has no sollution or has exactly one sollution or has infinitly many sollution.

$Ax = b$: 1, no, ∞

Proof:- If the system is inconsistant ~~then it~~ has no solution. #

Assume that the system $Ax = b$ is consistant system. #

Assume that $Ax = b$ has the two distinct solution x_1, x_2 So

حلين مختلفين

$Ax_1 = b \quad , \quad Ax_2 = b$

let $x_0 = x_1 - x_2$ then

$$Ax_0 = A(x_1 - x_2) = \underset{b}{Ax_1} - \underset{b}{Ax_2}$$

$$Ax_0 = b - b = \text{zero} \qquad (x_0 \neq 0).$$

Now, let $\bar{x} = x_1 + kx_0 \qquad k \in R$

$$\text{Now } A\bar{x} = A(x_1 + kx_0)$$

$$A\bar{x} = Ax_1 + Akx_0$$

$$A\bar{x} = b + k(Ax_0)$$

$$A\bar{x} = b + 0 = b$$

So $\bar{x}$ is a solution for $Ax = b$ $\forall k \in R$

So we have ∞ many solution #

Theorem 1.6.2

If A is an invertible $n\times n$ matrix then for each $n\times 1$ matrix b the system $Ax=b$ has exactly one sollution namely

$$x = A^{-1}b$$

تعليل $\rightarrow$ $Ax=b$

$A^{-1}Ax = A^{-1}b$

Example:- Solve

$$\begin{aligned} x+2y+3z &= 5 \\ 2x+5y+3z &= 3 \\ x \quad\;\; +8z &= 17 \end{aligned}$$

Sol: writ $Ax=b$

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}}_{b}$$

(we calculate A^{-1} befor)

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

the matrix is عكوس invertable

Now $x = A^{-1}b$

$$= \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

So. solution $\boxed{x=1}$, $\boxed{y=-1}$, $\boxed{z=2}$

طريقة لحل أنظمة بنفس مصفوفة المعاملات

Linear Systems with Common Coeficient matrix

To solve the system $Ax=b_1$, $Ax=b_2$, ..., $Ax=b_k$ we write $[A \mid b_1 \mid b_2 \mid \cdots \mid b_k]$ and solve together.

Example: Solve

(a)
$$\begin{aligned} x+2y+3z &= 4 \\ 2x+5y+3z &= 5 \\ x \quad\;\; +2z &= 9 \end{aligned}$$

(b)
$$\begin{aligned} x+2y+3z &= 1 \\ 2x+5y+3z &= 6 \\ x \quad\;\; +2z &= -6 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c|c} 1 & 2 & 3 & 4 & 1 \\ 2 & 5 & 3 & 5 & 6 \\ 1 & 0 & 2 & 9 & -6 \end{array}\right] \xleftrightarrow{\text{RREF}}$$

$$\left[\begin{array}{ccc|c|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

a) $x=1$, $y=0$, $z=1$

b) $x=2$, $y=1$, $z=-1$

Theorem 1.6.3

مصفوفة مربعة

If A is a square matrix, then

a) If B is a square matrix and $BA = I$ then $B = A^{-1}$.

b) If B is a square matrix and $AB = I$, then $\boxed{B = A^{-1}}$

Proof:-

a) Given that $AX = 0$ (zero)

$$\text{(}B\text{)}AX = B0 = 0$$

$$IX = 0$$

That mean $AX = 0$ has only trivial solution. So A is invertable. Now

$$BA = I$$

$$A^{-1}BA = A^{-1}I$$

$$BI = A^{-1}$$

$$B = A^{-1}$$

$I \Rightarrow$ كأنها مصفوفة / ذات المصفوفات

b) Use a) but with B insted (نزّل) of A.

$A = B^{-1}$ and so

$$\boxed{A^{-1} = B}$$

هذه النظرية ستكرر على مدار الفصل مع زيادة في

Theorem 1.6.4

If A is $n \times n$ matrix then the following statement are Equevellent (T.F.S.E)

a) A is invertable

b) $Ax = 0$ has only the trivial solution

c) The Reduced Row Echlone form of A is the I_n

d) A is expressible as a product of elemantary matrix.

e) $Ax = b$ is Consistant for $\forall$ $n \times 1$ matrix b.

f) $Ax = b$ has exactly one solution for every $n \times 1$ matrix b.

$a \to b \to c \to d \to a$ اليوم بدرسنا

$a \to f \to e \to a$

Th 1.5.3: $a \to b \to c \to d$

$a \to e \to$ Th 1.6.2

Proof:- $a \to f$: Theorem 1.6.2 مرجوعي

$f \to e$: Assume $Ax = b$ has exactly one solution $\forall$ $n \times 1$ b matrix.

So $Ax = b$ is Consistant.

$e \to a$:- Assume that $Ax = b$ is Consistant for $\forall$ b.

ناتج ↓

(a) A is invertable.

$AB = I$ لازم نثبت

ناتج → That mean every system of the following have solation

$$Ax_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ Ax_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ Ax_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix},\ Ax_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}_{n-1},\ Ax_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}_n$$

So, let x_1 be $\ni Ax_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

x_2 be $\ni Ax_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$

x_3 be $\ni Ax_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$

$\vdots$

x_{n-1} be $\ni Ax_{n-1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}$

x_n be $\ni Ax_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

$$[Ax_1 | Ax_2 | Ax_3 | \cdots | Ax_{n-1} | Ax_n] = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = I$$

$$A[x_1 | x_2 | x_3 | \cdots | x_{n-1} | x_n] = I$$

let $B = [x_1 | x_2 | \cdots | x_n]$, So be is $n \times n$ matrix and

$AB = I$, So is invertable

لازم نحفظ

Theorem 1.6.5

> let A and B a sequare matrices of same size. If AB is invertable, then A and B are also invertable.

اذا كانت مصفوفتين مربعتين من نفس الرتبة وكانت AB لها معكوس فإن A و B لها معكوس. اثباتها في الامتحان

When we can say (the system is Consistant)?

$$Ax = b$$

A invertable → b

A Condition or b $\ni Ax = b$ is consis.

ما هي قيم b التي تجعل النظام قابلاً للحل توضيح أو ما الشروط التي يجب أن تحققها أي مصفوفة b حتى يجعل النظام قابلاً للحل

We Ask about the values of b in which $Ax = b$ is Consistant for a fixed matrix b.

Example:- What Condition must b have in which the following systems are Consistant. $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

لاحقاً

سؤال امتحان

a) حاس اشروط

$$x + y + 2z = b_1$$
$$x + z = b_2$$
$$2x + y + 3z = b_3$$

Sol. the A.M. is (augmented matrix)

$(-1)^{i+j}$

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 1 & 0 & 1 & b_2 \\ 2 & 1 & 3 & b_3 \end{bmatrix}$$

GE

$R_2 : R_2 - R_1$
$R_3 : R_3 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & -1 & -1 & b_2 - b_1 \\ 0 & -1 & -1 & b_3 - 2b_1 \end{bmatrix}$$

$R_2 : -R_2$
$R_3 : R_3 + R_2$

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix}$$

$$b_3 - b_2 - b_1 = 0$$
$$b_3 = b_2 + b_1$$

$b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

So, we must have

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_1 + b_2 \end{bmatrix} \Rightarrow b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \checkmark$$

b) $x + 2y + 3z = b_1$
$2x + 5y + 3z = b_2$
$x + 8z = b_3$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

we know that A is invertable so, we have no condition on b to give the system consistance.

$$X = A^{-1} b = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ So:

$$X = -40b_1 + 16b_2 + 9b_3$$
$$y = 13b_1 - 5b_2 - 3b_3$$
$$z = 5b_1 - 2b_2 - b_3$$

no cons

The end of the section

الصفوفات القطرية والصفوفات شبه القطرية والصفوفات المتماثلة

Sec (7) ~~Digital~~, Trigonal and Symmetric Matrices:-

Diagonal, Trigonal and " = :-

Diagonal matrices:-

def: A square matrix is wich all the entries off the main diagonal are zero's is called a diagonal.

المصفوفة التي جميع عناصرها خارج القطر الرئيسي أصفار تسمى مصفوفة قطرية. بغض النظر عن عناصر القطر الرئيسي

Examples:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Remark:- If $D = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$, then

a) $d_i \neq 0 \ \forall i$, then D is invertable

and $D^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{d_2} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{d_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{d_n} \end{bmatrix}$

b) $D^k = \begin{bmatrix} d_1^k & 0 & 0 & \cdots & 0 \\ 0 & d_2^k & 0 & \cdots & 0 \\ 0 & 0 & d_3^k & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n^k \end{bmatrix}$

Example:- $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$, then

→ Diagonal
→ invertable
A is invertable
لأن القطر الرئيسي لا يحتوي أصفار

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{-1}{3} \end{bmatrix}$$

$$A^{3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -27 \end{bmatrix}$$

$$A^{-3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{-1}{27} \end{bmatrix}$$

تم بعون

Trigonal matrices :-

def:- A square matrix with zero entries in all position above the main diagonal is called ~~trigonal matrices~~ a lower trignal matrix

المصفوفة المربعة التي جميع المواقع أعلى القطر الرئيسي أصفار تسمى a lower trigonal matrix.

def:- " " " "
below
called upper trigonal matrix

A matrix that is either upper or lower trigonal is called trigonal matrix.

a) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ U.T

b) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 9 & 5 \end{bmatrix}$ L.T

c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ U.T & diagnal matrix
L.T

Theorem 1.7.1

a) The Transpose of a lower (upper) trigonal matrix is upper (lower) trigonal.

b) The product of lower (upper) trigonal matrices is lower (upper) trignalar matrix.

$$\boxed{\begin{array}{l} \text{Lower} \times \text{Lower} = \text{Lower} \\ \text{upper} \times \text{upper} = \text{upper} \end{array}} \longleftarrow \text{المقصد (b)}$$

c) A trigonal matrix is invertable if and only if (iff) its diagonal entries are non zeros.

d) The inverse of an invertable lower (upper) tringonal matrix is lower (upper) trigonal matrix.

(معكوس اللور لور ومعكوس الأبر أبر)

Example: let $A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

then $A^{-1} = \begin{bmatrix} 1 & \frac{-3}{2} & \frac{7}{5} \\ 0 & \frac{1}{2} & \frac{-2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$, B is not invertable becase the main diagonal contain zero

$$AB = \begin{bmatrix} 3 & -2 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

upper . upper = upper

Symmetric Matrices:-

هي المصفوفة التي تساوي مدورها

A square matrix is called symmetric iff

$$A^T = A$$

تسمى المصفوفة متماثلة إذا كان مدورها هو نفسها

Example:

a) $\begin{bmatrix} 7 & -3 \\ -3 & 5 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 4 & 5 \\ 4 & -3 & 0 \\ 5 & 0 & 7 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ sym. هي diagonal is

كل مصفوفة قطرية
هي مصفوفة متماثلة

Theorem:- 1.7.2

If A and B are symmetric matrices with same size if k is any scalar, then:-

a) A^T is symmetric.

b) $A \pm B$ is symmetric.

c) kA is symmetric.

The proof a + c Exersise:-

$\because$ A is symmetric

$\therefore$ A^T is symetric

$$(A \pm B)^T = A^T \pm B^T = A \pm B \Rightarrow A \pm B \text{ is sym}$$

$$(kA)^T = kA^T = kA \Rightarrow kA \text{ is sym}$$

Proof (b)

$$(A \pm B)^T = A^T \pm B^T = A \pm B$$

So $A \pm B$ is symmetric.

Remark:- If A and B are symmetric, then it is not necessary that AB is symmetric.

Example:- a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix}$

$AB = \begin{bmatrix} -2 & 1 \\ -5 & 2 \end{bmatrix}$ not sym.

b) $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -4 & 3 \\ 3 & -1 \end{bmatrix}$

$AB = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ is sym.

Theorem 1.7.3

If A is an invertable symmetric matrix then A^{-1} is symmetric.

Proof:- ~~Example~~ Ex:- Assume A is sym & inver.

$$(A^{-1})^T = (A^T)^{-1} = \boxed{A^{-1}}$$

So A^{-1} is symmetric,

Example:- If $A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}$, then 2×3

$A^T = \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix}$ 3×2

$AA^T = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 10 & -2 & -11 \\ -2 & 4 & -8 \\ -11 & -8 & 41 \end{bmatrix}$ is Symm.

$A^TA = \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}$

$= \begin{bmatrix} 21 & -17 \\ -17 & 34 \end{bmatrix}$

حول القطر الرئيسي
متناظر

Theorem:- 1.7.4

If A is an invertable matrix then AA^T and A^TA are invertable.

Proof:- Ex:

$$(AA^T)^T = A^T (A^T)^T = A^T . A$$

مثال توضيحي

الحددات

Chapter(2) Determinants

sec(1) Determinante by Cofactor expansion.

def:- The determinant of a square matrix A denoted by det(A) or |A| is a function determinate on the set of all square a matrix to the set of real numbers if $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is any 2x2 matrix

then $\det(A) = ad - bc$

Example:- a) $A=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\det(A) = 4 - 6 = -2$

b) $A=\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ det(A) = no det.

c) $A=\begin{bmatrix} 2 & -1 & 0 \\ 4 & 3 & 1 \\ -2 & 5 & 6 \end{bmatrix}$ det(A) = is difficult to calculate now.

ايجاد المحددات باستخدام cofactor

def:- If A is a square matrix, than the minor of the entry a_{ij} denoted by M_{ij} is the is the determinants of the submatrix remains when we delete the row i and the column j from A. The number $(-1)^{i+j} M_{ij}$ denoted by C_{ij} is called the Cofactor of the entry a_{ij}

العامل

$= (-1)^{i+j} M_{ij}$

Example:- $A=\begin{bmatrix} 2 & -1 & 0 \\ 4 & 3 & 1 \\ -2 & 5 & 6 \end{bmatrix}$

$M_{11} = \begin{vmatrix} 3 & 1 \\ 5 & 6 \end{vmatrix} = 18 - 5 = 13$

$C_{11} = (-1)^{1+1} M_{11} = (-1)^2 . 13 = 13$

$M_{23} = \begin{vmatrix} 2 & -1 \\ -2 & 5 \end{vmatrix} = 10 - 2 = 8$

$C_{23} = (-1)^{2+3} M_{23} = -8$

Cofactor expresion :-

a) If A is 3x3 matrix then $\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

b) If A is nxn matrix, then
$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$

this method is called Cofactor expansion along the first row of A.
طريقة حساب المحدد على السطر الأول للمحددات .

Example: $A = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 3 & 1 \\ -2 & 5 & 6 \end{bmatrix}$

Find det(A)

Cofactor

$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

$= 2\begin{vmatrix} 3 & 1 \\ 5 & 6 \end{vmatrix} - (-1)\begin{vmatrix} 4 & 1 \\ -2 & 6 \end{vmatrix} + 0\begin{vmatrix} 4 & 3 \\ -2 & 5 \end{vmatrix}$

$= 2 \times 13 + 1(24+2) + 0$

$26 + 26 = 52$

Theorem 2.1.1

The determinant of an nxm matrix A can be computed by Multiplying the entries in any row and any Column by their Cofactors and adding the resulting products, that is for each $1 \le i \le n$, or each $1 \le j \le n$, we have :-

نجمع توزيع الضرب

$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$

هذه الطريقة حساب المحدد باستخدام الصف i

$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$ entry

a_{ij}. طريقة حساب المحدد باستخدام (j)

هنا لو بدلنا بأي عمود أو صف

Example: Find det(A) if

$A = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 3 & 1 \\ -2 & 5 & 6 \end{bmatrix}$

باستخدام العمود الثاني

$\det(A) = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$

$= (-1)(-26) + 3(-1)^4\begin{vmatrix} 2 & 0 \\ -2 & 6 \end{vmatrix} + 5(-1)^{3+2}\begin{vmatrix} 2 & 0 \\ 4 & 1 \end{vmatrix}$

$= 26 + 36 - 10 = 52$

b) $A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$

ندور على الأصفار الأكثر

حساب محدد لـ 4x4 باستخدام العامود الثاني

Along the second Colum.

$$\det(A) = a_{22} C_{22}$$

$$= 1(-1)^4 \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix}$$

$$= 1\left[-2(-1)^4 \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}\right] = (1)(-2)(1)(3) = -6$$

Adjoint of a matrix:- هي مدور مصفوفة المعاملات

def- If A any nxn matrix and C_{ij} the cofactor of a_{ij} then the matrix.

بدل كل عنصر نحط الـ cofactor تبعه

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & \cdots & C_{1n} \\ C_{21} & C_{22} & C_{23} & \cdots & C_{2n} \\ \vdots & & & & \\ C_{n1} & C_{n2} & \cdots & & C_{nn} \end{bmatrix}$$

is called Cofactor matrix of A.

The transpose of this matrix is called

Co-factor مدور الـ

the adjoint matrix of A and we write

$$\text{adj}(A) = C^T \rightarrow \text{cofactor مدور مصفوفة}$$

Example: Find adj(A) if $A = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 3 & 1 \\ -2 & 5 & 6 \end{bmatrix}$

$C_{11} = 13$, $C_{12} = \begin{vmatrix} 4 & 1 \\ -2 & 6 \end{vmatrix} = -26$, $C_{13} = 26$

$C_{21} = -6$, $C_{22} = 12$, $C_{23} = -8$

$C_{31} = -1$, $C_{32} = 2$, $C_{33} = 10$

$$\text{adj}(A) = \begin{bmatrix} 13 & -26 & 26 \\ -6 & 12 & -8 \\ -1 & 2 & 10 \end{bmatrix}^{T}$$ مدور

المصفوفة الأساسية

$$\text{adj}(A) = \begin{bmatrix} 13 & -6 & -1 \\ -26 & 12 & 2 \\ 26 & -8 & 10 \end{bmatrix}$$

الأسئلة المطلوبة

(1 — 19) odd sec(2), 20, 22, 25, 27, 28, 31

sec(3) ⇒ 1, 2, 5, 3, 8, 12, 13, 14, 19, 20, 22, 25, 27, 29

sec(4) ⇒ 1→15 odd, 16, 17, 20, 21, 23, 24, 25, 27, 29, 30, 31, 34, 35

30

sec(5) ⇒ 1, 3, 5, 6, 9, 10, 11, 14, 16, 19, 22

sec(6) ⇒ 1 →(odd) 21, 24, 27, 28, 30

sec(7) ⇒ 1 — 16, 18, 19, 22, 30

---

sec(2) أخذ (1) 1 → 23 odd, 24, 24, 37, 35

sec2.2 (2) 1 — 11 odd, 12, 13, 20, 21

sec(2.3) ⇒ , 4, 5, 6, 7, 9, 11, 12, 13, 16, 18, 20, 22, 23

sec(2.4) ⇒ 1 — 13 odd, 18, 20, 21, 23

اذا كان لدينا مصفوفة مثلثية تحتية علوية أو سفلية
فلإيجاد محدد هذه المصفوفة نضرب عناصر القطر الرئيسي

مهم

أهم إشي هون

Theorem (2.1.2)

If A is invertable (قابلة للعكس / أي معرفة المعكوس), then

$$\boxed{A^{-1} = \frac{1}{\det(A)} \cdot adj(A)}$$

مهم جدا — محدد المصفوفة

Example:

If $A = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 3 & 1 \\ -2 & 5 & 6 \end{bmatrix}$, then

$$A^{-1} = \frac{1}{\det(A)} \cdot adj(A) = \frac{1}{52} \begin{bmatrix} 13 & -6 & -1 \\ -26 & 12 & 2 \\ 26 & -8 & 10 \end{bmatrix}$$

محدد المصفوفة السابقة

$$A^{-1} = \begin{bmatrix} \frac{1}{4} & -\frac{3}{26} & -\frac{1}{52} \\ -\frac{1}{2} & \frac{3}{13} & \frac{1}{26} \\ \frac{1}{2} & -\frac{2}{13} & \frac{5}{26} \end{bmatrix}$$

Theorem :- (2.1.3) (upper / lower Diagonal)

If A is $n \times n$ trigonular matrix, then

$\det(A) = a_{11} a_{22} a_{33} \cdots a_{nn}$ محددها يساوي ضرب عناصر القطر الرئيسي

Example:-

let $A = \begin{bmatrix} 2 & 7 & -3 & 8 & 3 \\ 0 & -3 & 5 & 7 & 1 \\ 0 & 0 & 6 & 7 & 6 \\ 0 & 0 & 0 & 9 & 8 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$ upper trigonal

then

$\det(A) = 2 \cdot (-3)(6)(9)(4) = \boxed{-1296}$

الشروط: عدد المعادلات يساوي n
← عدد المجاهيل يساوي n
← محددها ≠ صفر

كاعدة كرامر

Cramer's Rule:

مصفوفة المعاملات مربع ومحدد لا يساوي صفر

Theorem (2.1.4): If $AX=b$ is any system of n linear equation on n unknowns such that $\det(A) \neq 0$, then the system has a uniqe solution and this solution is :-

$$x_1 = \frac{\det(A_1)}{\det(A)},\ x_2 = \frac{\det(A_2)}{\det(A)},\ \ldots\ldots,\ x_n = \frac{\det(A_n)}{\det(A)}$$

where A_j is the matrix obtained by replacing the entries in the i_{th} Column of A the matrix $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

مهم جداً ولازم نتدرب

Example: Solve the system by Using Cramer Rule:

$2x - y = 3$
$4x + 3y + z = 3$
$-2x + 5y + 6z = 5$

$$AX = b \Rightarrow \begin{bmatrix} 2 & -1 & 0 \\ 4 & 3 & 1 \\ -2 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 3 & 1 \\ -2 & 5 & 6 \end{bmatrix},\ A_1 = \begin{bmatrix} 3 & -1 & 0 \\ 3 & 3 & 1 \\ 5 & 5 & 6 \end{bmatrix},\ A_2 = \begin{bmatrix} 2 & 3 & 0 \\ 4 & 3 & 1 \\ -2 & 5 & 6 \end{bmatrix}$$

بأقيم العمود الأول وأضع الثوابت

$\times 13$, 3, 39 $\quad$ $+13$ $\quad$ $+13$, 39, 52

حل الأنظمة باستخدام كرامر

$$A_3 = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 3 & 3 \\ -2 & 5 & 5 \end{bmatrix}$$

تعالوا نحسب محدد كل مصفوفة

$\det(A) = 52$, $\det(A_1) = 52$, $\det(A_2) = -52$
$\det(A_3) = 104$

$$x = \frac{\det(A_1)}{\det(A)} = \frac{52}{52} = 1$$

$$y = \frac{\det(A_2)}{\det(A)} = \frac{-52}{52} = -1$$

$$z = \frac{\det(A_3)}{\det(A)} = \frac{104}{52} = 2$$

The Solution
$(x, y, z) \Rightarrow (1, -1, 2)$

سؤال: أنظر إليها
مهم
See: Ex (2+3+6+7) sec(1)

Solve the system:- الحل النهائي أنه يطلع
$(0, 1, 1)$

$3x + y = 1$
$-2x - 4y + 3z = 1$
$5x + 4y - 2z = +2$

صـــــا بأحمد بالاقتران الصف

sec(2) Evalution Determination by Row Reduction :-

Theorem (2.2.1) let A be a square matrix with a row of zero's or a column of zero's then

1

$$\det(A) = 0$$

لو كان عندي صف صفري في مصفوفة الناتج صفر
لأن اصفار ... أحمد ... صفر

(اثبت النظرية)

Theorem :- (2.2.2)

2

let A be a square matrix then $\det(A) = \det(A^T)$

Theorem (2.2.3) ا، مع الإثبات

3

let A be $n \times n$ matrix, then

a) If B is the matrix results when a single row or a single Column k, then

$$\det(B) = k \det(A)$$

$A \xrightarrow[C_j \to kC_j]{R_i \to kR_i} B$

$\det(B) = k(\det(A))$

b) If B is the matrix results ~~when a single~~ two rows or columns of A are interchang

بدلنا

then $\det(B) = -\det(B)$

$A \xrightarrow[C_i \Leftrightarrow C_j]{R_j \Leftrightarrow R_i}$

c) If B ---------- when a multiple of a one row or column of A is added to another row or Column, then.

$$\det(A) = \det(B)$$

Theorem (2.2.4) :- مصفوفة الوحدة

let E be $n \times n$ elementary matrix, then

a) If E is results from I_n by multiplying a row of I_n by k, then $\det(E) = k$

b) If ---------- interchanging two rows then $\det(E) = -1$

b) If ---------- by adding a multiple of a row of I_n to another raw, then

$$\det(E) = 1$$

نفس النظرية السابقة

Tow >

$I_n \xrightarrow{\text{ضرب } k} E \qquad \det E = k$

$I_n \xrightarrow{\text{تبديل}} E \qquad \det E = -1$

$I_n \xrightarrow{\text{ضرب وجمع}} E \qquad \det E = 1$

Example: جزء المصفوفة الثانوية

a) $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 2$ $\underline{K}$ ضرب الصف الثاني في 2

بدل السطر الثاني بالثالث

b) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -1$ بدل الصفوف

جزء السطر الثاني في ٣ ثم جمعه

c) $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1$ ضرب الصف الأول في ٣ واجمعه للصف الثالث

cause

Theorem (2.2.5) If A is a square matrix with two propotional rows or columns, then $\det(A) = 0$

a) $\begin{vmatrix} 2 & 1 & 3 \\ 1 & 2 & 4 \\ 6 & 3 & 9 \end{vmatrix} = 0$ الصف الأول والثالث propotional, $R_3 = 3R_1$

b) $\begin{vmatrix} 1 & -2 & 7 \\ -4 & 8 & 5 \\ 2 & -4 & 3 \end{vmatrix} = 0 \longleftrightarrow$ propotional عمودين

$C_2 = -2C_1$

row red باستخدام

Examples:- Calculate the following determinites :- محددات

ERO باستخدام

[1] $\begin{vmatrix} 2 & -1 & 0 \\ 4 & 3 & 1 \\ -2 & 5 & 6 \end{vmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{matrix} \quad \begin{vmatrix} 2 & -1 & 0 \\ 0 & 5 & 1 \\ 0 & 4 & 6 \end{vmatrix}$

$R_3 \to R_3 - \frac{4}{5}R_2 \quad \begin{vmatrix} 2 & -1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & \frac{26}{5} \end{vmatrix} = 2 \cdot 5 \cdot \frac{26}{5} = 52$

Trigonal upper

[2] $\begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & -5 & 1 \end{vmatrix} = -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & -5 & 1 \end{vmatrix} = -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & -5 & 1 \end{vmatrix}$

$R_3 \to R_3 - 2R_1 \quad = -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & -1 & -5 \end{vmatrix} = 3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 1 & 5 \end{vmatrix} = 3 \cdot (0) = \boxed{0}$

المحدد بيساوي صفر

3
$$\begin{vmatrix} 1 & 0 & 0 & 3 \\ 2 & 7 & 0 & 6 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -5 \end{vmatrix} \xrightarrow{C_4 \to C_4 - 3C_1} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 7 & 0 & 0 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -26 \end{vmatrix}$$

$$= 1.7.3.(-26) = \boxed{-546} \ \#$$

4
$$\begin{vmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & -1 & 1 \\ 3 & 5 & -2 & 6 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{vmatrix}$$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - 2R_1$
$R_4 \to R_4 - 3R_1$

$$(-)\begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & -1 & +1 & 3 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix} \underset{R_3 \to \frac{1}{3}R_3}{\xrightarrow{R_4 \to R_4 + R_2}} = (-3) \begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & \frac{3}{3} & \frac{3}{3} \\ 0 & 0 & 9 & 3 \end{vmatrix}$$

$$R_4 \to R_4 - 9R_3 = (-3) \begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -6 \end{vmatrix} = (-3)(1)(-1).(1).(-6) = \boxed{-18} \ \#$$

The End of the sec(2)

جميع الحقوق محفوظة

sec(3) Properties of the determinetes function

Remark:- If A is $n \times n$ matrix ~~...~~ and k is any scalar then $\det(kA) = k^n \det(A)$

$$|kA| = \begin{vmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ k & & & \\ & & & \\ ka_{n1} & k & & \end{vmatrix}$$

$\det(k(A)) = k^n \det(A)$

Example:-

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 3 & 1 \\ -2 & 5 & 6 \end{bmatrix}, \ \det(A) = 52$$

$$\det(2A) = \begin{vmatrix} 4 & -2 & 0 \\ 8 & 6 & 2 \\ -4 & 10 & 12 \end{vmatrix} = (2^3).(52) = \boxed{}$$

Example: $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

$\det(A) = 5 - 4 = \boxed{1}$, $\det(B) = 9 - 1 = \boxed{8}$

$A + B = \begin{bmatrix} 4 & 3 \\ 3 & 8 \end{bmatrix}$, $\det(A+B) = \boxed{23}$

$\det(A+B) = 23 \neq \boxed{9} = \det(B) + \det(A)$

Remark: In the general

$$\det(A+B) \neq \det(A) + \det(B)$$

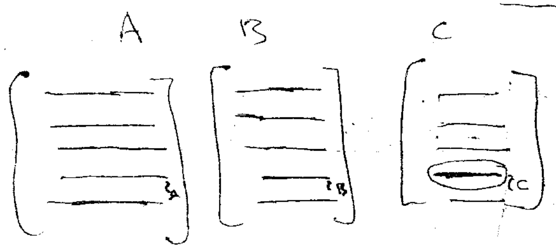
Theorem (2.3.1)

Let A, B, C be $n \times n$ matrices, that differ only on a single row (say in the i_{th} row) and assume that the ith row of C can be obtained by adding corresponding entries in the i_{th} row of A and B, then

$$\det C = \det(A) + \det(B)$$

The same results hold for columns.

توضيح النظرية



$\boxed{r_C = i_A + i_B}$

Example:
$$A = \begin{bmatrix} 2 & -1 & 0 \\ 2 & 2 & 0 \\ -2 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 & 0 \\ 2 & 1 & 0 \\ -2 & 5 & 6 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 3 & 0 \\ -2 & 5 & 6 \end{bmatrix}$$

هذا الصف حاصل جمع الصفين

نجمع الصفين يساوي الـ A,B

$\det(A) = 36$ 24

$\det(B) = 24$ 28

$\det(C) = 60$ 52

$\det(C) = 60$

$$\boxed{\begin{gathered}\det(C) = \det(A) + \det(B) \\ 36 + 24\end{gathered}}$$ #

أمور نستخدم قبل عملية الإثبات

lemma (2.3.2)

If B is $n\times n$ matrix and E is an elementary matrix then:-

$$\boxed{\det(EB) = \det(E).\det(B)}$$ — ⓐ المحدد يتوزع على عملية الضرب

Note/
we can generalized this for a finite number elementary matrices, that is

$$\det(E_1.E_2.E_3 \dots E_n B) = \det(E_1).\det(E_2).\det(E_3) \dots \det(E_n)\det B$$

Theorem:- (2.3.3)

A matrix A is invertable iff $\det(A) \neq$ zero — ⓑ
صفر

أفضل طريقة لحساب اذا المصفوفة قابلة للعكس أم لا، وبالتالي بأحسب المحدد

Proof:- Asume that A is invertable, then:- الشرط الطرفين
we can write $A = E_1.E_2.E_3 \dots E_n$ where E_i is an Elementary matrix $\forall i$.

Now, $\det(A) = \det(E_1 E_2 E_3 \dots E_n) = \det(E_1).\det(E_2) \dots \det(E_n)$

But $\det(E_i) \neq 0 \ \forall i$

So $\det(A) \neq 0$ ضرب أعداد غير صفرية

$B \xrightarrow{RREF}$ I_n → A is invert.
row of zero ✗ مستحيل

assume $\det(A) \neq 0$
now, If B is the RREF of A, then:-

$B = E_1 E_2 E_3 \dots E_L A$

$\det(B) = \det(E_1)\det(E_2).\det(E_3) \dots \det(E_L).\det(A) \neq \boxed{0}$

So $B = I_n$

$(E_1 E_2 \dots E_L)A = I_n$

So A is invertable and $A^{-1} = (E_1 E_2 \dots E_L)$

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 4 & 6 & 10 \end{bmatrix}$ $= \det(A) = 1((2\times(10)-24) - 2(20-16) + 3(12-8))$
$= \boxed{0}$

$\det(A) = 0$, So A is not invertable.

Theorem (2.3.4) If A and B are square matrixes of the same size, then

$$\boxed{\det(AB) = \det(A).\det(B)}$$ — ⓒ

Proof:-
If A is invertable, then we can write
$A = E_1.E_2 \dots E_k$ where E_i is Elementary matrix $\forall i$.

Now, $\det(AB) = \det(E_1 E_2 \dots E_k B) = \det(E_1) \dots \det(E_k).\det(B)$
$= \det(E_1 E_2 \dots E_k).\det(B)$
$= \det(A).\det(B)$ ← invertable فرضية

Now, If A is not invertable ... $\det(A) = 0$
then AB is not invertable نظرية سابقة
$\det(AB) = 0$
$= 0.\det(B)$
$= \det(A).\det(B)$

$\det(AB) = \det(A).\det(B)$ #

Example:- $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 3 \\ 5 & 8 \end{bmatrix}$

$$AB = \begin{bmatrix} 2 & 17 \\ 7 & 14 \end{bmatrix}$$

$\det(AB) = -23$

$\det(A) = 1$

$\det(B) = -23$

Theorem (2.3.5):- If A is invertable then $\boxed{5}$

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

دي كمان ممكن تيجي في الامتحان
الاثبات علي السريع

Proof:- Since A is invertable then $\det(A) \neq 0$ and $AA^{-1} = I$

Now:-

$$\det(AA^{-1}) = \det(I)$$

$$\det(AA^{-1}) = 1$$

$$\det(A).\det(A^{-1}) = 1$$

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

EM
B
X , ✓

نظرية سابقة

Theorem (2.3.6) - راجع للإثبات

A is invertable (b → f)

$\det(A) \neq 0$

End of sec(3)

Sec(4):- A combinational Approach of Determinates:-

الوصول للمحددات باستخدام التوافيق

هذه الطريقة تستخدم للمحددات الصغيرة (2×2) (3×3) أما الأكبر فلا لأنها معقد.

def:- A permutation (تبديل) of the set of integers $\{1, 2, 3, \ldots, n\}$ is an arragment (ترتيب) of these integers in some (بعض) order without omissions (حذف) or repetitions.

بدون حذف أو تكرار

Examples:

① $A = [1, 2, 3]$

$3! = 6$

1	2	3
2, 3	1 3	1 2
3 2	3 1	2 1

(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2)
(3,2,1)

(2) $A = \{1, 2, 3, 4\}$ $\qquad 4! = 24$

~~1 2 3 4~~

1 → 2, 3, 4 ; 2 → 1, 3, 4 ; 3 → 1, 2, 4 ; 4 → 1, 2, 3

2 3 4
3 4
4 3

$24 = 4!$ **Remark:**

The number of Permations on a set $\{1, 2, 3, \dots, n\}$ is $n!$ ⊛

يقال انه قد حدث انقلاب

def. An **inversion** is said to be occure in a permotation $(j_1, j_2, j_3, \dots, j_n)$ when a larger integer precedes (يسبق) a smaller one.

يقال انه قد حدث انقلاب في التبديل اذا سبق عدد كبير واحد صغير

If the totall number of **inversion** in a pormotation is even (odd) then we say the pormotation is even (odd).

Example: (a) $(\boxed{7}, 2, 1, 5, 4, 3, 6)$

$$6 + 1 + 2 + 1 + 0 + 0 = \boxed{10} \leftarrow \text{even}$$

So the pormotation is even.

(b) $(6, 2, 3, 1, 4, 5)$

$$5 + 1 + 1 + 0 + 0 + 0 = \boxed{7} \leftarrow \text{odd}$$

So a pormotation is odd

(c) $(1, 2, 3, 4)$

$$0 + 0 + 0 + 0 = \boxed{0} \leftarrow \text{even}$$

So a pormotatio is even.

(d) Classify the permotation on $\{1, 2, 3\}$

يتبع

classify

	inversion	
$(1,2,3)$	$\to 0$	even
$(1,3,2)$	$\to 1$	odd
$(2,1,3)$	$\to 1$	odd
$(2,3,1)$	$\to 2$	even
$(3,1,2)$	$\to 2$	even
$(3,2,1)$	$\to 3$	odd

odd عدد = even عدد

Def. [I] If A is $n \times n$ matrix, then we mean by an elementary product from the product of (n) entries from A, No two of then come from the same rows the same column. In other word

$a_{1j_1} \cdot a_{2j_2} \cdot a_{3j_3} \cdot \ldots a_{nj_n}$, where

$j_i \neq j_k \quad \forall \; i,k \leq n$

[2] A single Elementary product from $n \times n$ matrix A is elementary product $a_{1j} a_{2j} a_{3j} \ldots a_{nj_n}$ multiplied by $\boxed{1}$ or $\boxed{-1}$ we used $\boxed{1}$ it the permotation $(j_1, j_2, j_3, \ldots j_n)$ is <u>even</u> Peromotation and used $\boxed{-1}$ if it is <u>odd</u>.

Remark: If A is <u>$n \times n$ matrix</u>, the A has $n!$ elementary product.

Example

(a) $A = \begin{bmatrix} a_{11} & \boxed{a_{12}} \\ \boxed{a_{21}} & a_{22} \end{bmatrix}$

	Permotation	Inversion		S.E.P
$a_{11} a_{22}$	$(1,2)$	0	even	$a_{11}a_{22}$
$a_{12} a_{21}$	$(2,1)$	1	odd	$-a_{12}a_{21}$

$\det(A) = a_{11}a_{22} - a_{12} \cdot a_{21}$

(b) $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$

(c) $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

E.P

	Por.	inversion
$a_{11} a_{22} a_{33}$	$(1,2,3)$	0
$a_{11} a_{23} a_{32}$	$(1,3,2)$	1
$a_{12} a_{21} a_{33}$	$(2,1,3)$	1
$a_{12} a_{23} a_{31}$	$(2,3,1)$	2
$a_{13} a_{22} a_{31}$	$(3,2,1)$	2
$a_{13} a_{21} a_{32}$	$(3,1,2)$	3

even	
odd	

E - Product

Def:- let A be a square matrix. we define $\det(A)$ to be the sum of all singhed elementary product from A.

Example:

(a) $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$

(b) $A = \begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = -6 - 4 = \boxed{-10}$ #

$$B = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 3 & 1 \\ -2 & 5 & 6 \end{bmatrix} \begin{matrix} 2 & -1 \\ 4 & 3 \\ -2 & 5 \end{matrix}$$

$\det(B) = (36 + 2 + 0) - (0 + 10 - 24)$

$38 + 14 = 52$

The End of Chapter (2).

Chapter (5) General Vector spaces :-

Sec (5.1) ~~Real~~ vector spaces:-

def:- Given that V is a non empty set We define two operations on V, one is called (addition) denoted by $(+)$ and associating with a pair of element u and v of V. The other is called a scallar multiplication denoted by $(\cdot)$ and associating with an element v of V and scaller element (k) (here is $\mathbb{R}$)
If the following ten axioms hold for all $u, v, w \in V$ and scalars k, m then we called V a vector space and the element V is called a vector.

5-1

1. $u + v \in V \quad \forall u, v \in V$
(Closed under addition)
2. $u + v = v + u$ (commutitive)
3. $u + (v + w) = (u + v) + w$ (associative)
4. There is an element $0 \in V$ called
(zero or Indentity) vector such that
$u + 0 = 0 + u = u \quad \forall u \in V$
5. For each $u \in V$, there is an element

usual addition

(-u) Called (negative or inverse) of u such that

$u+(-u)=0$

مغلقة على عملية الضرب

⑥ $ku \in V \ \forall u \in V$ (closed under scalar multiplication) $\forall k \in \mathbb{R}$

(د.ط) للضرب

⑦ $k(u+v) = ku + kv$

التوزيع (Distribution)

⑧ $(k+m)u = ku + mu$

⑨ $k(mu) = (km)u$

⑩ $1.u = u$

Vector space = عشرة شروط + العمليتين + المجموعة

لازم كله يتحقق عشان نقول (vector space)

(الجمع والضرب)

حالية

حتى نحفظ عشر شروط أول خمسة على عملية الجمع وتاني خمسة على الضرب

١- لإثبات أن المجموعة V-space مع العمليات المعطاة يجب أن نثبت جميع الـ ١٠ شروط لجميع العناصر من V وجميع الثوابت الممكنة

٢- لا يمكن أن يمثل شرط واحد عنصر واحد

٣- نستطيع أن نعرف عمليات كما نشاء ولكن حتى تكون لدينا V-space يجب تحقق الشروط السابقة

$(+, \cdot)$

Example: ①

العمليات الفعلية

① Take $V = \mathbb{R}$ with usuall operation.

العمليات العادية (الجمع والضرب)

Sol:

① $u+v \in \mathbb{R} \quad \forall u,v \in \mathbb{R}$

② $u+v = v+u \quad \forall u,v \in \mathbb{R}$

③ $u+(v+w) = (u+v)+w \quad \forall u,v,w \in \mathbb{R}$

④ $0 \in \mathbb{R}$ and $u+0 = 0+u = u \quad \forall u \in \mathbb{R}$

⑤ If $u \in \mathbb{R}$ then $-u \in \mathbb{R}$ and $u+(-u) = 0$

⑥ $ku \in \mathbb{R} \quad \forall u \in \mathbb{R}, k \in \mathbb{R}$

⑦ $k(u+v) = ku+kv \quad \forall u,v \in \mathbb{R}, k \in \mathbb{R}$

⑧ $(k+m)u = ku + mu \quad \forall k,m \in \mathbb{R}$ & $u \in \mathbb{R}$

⑨ $k(mu) = (km)u \quad \forall u \in \mathbb{R} \quad k,m \in \mathbb{R}$

⑩ $1.(u) = u \quad \forall u \in \mathbb{R}$

العمليات = العادية

$(\mathbb{R}, +, \cdot)$ is Vector space with st operation

جمع ضرب

٤- تغيير المجموعة وتثبيت العمليات أو تثبيت المجموعة وتغيير العمليات قد يغير حالة المجموعة من v-space إلى not v-space والعكس

٥- العنصر الصفري ليس بالضرورة أن يكون هو الصفر العادي وإنما يرتبط شكله وقيمته بطبيعة عناصر المجموعة والعمليات المعرفة عليها

$R^n = \{(x_1, x_2, \dots x_n) ; x_i \in \mathbb{R}\}$ $(\mathbb{R}^n, +, \cdot)$

Example ② let $V = \mathbb{R}^n$, $n \in \mathbb{N}$. Define

$$x + y = (x_1, x_2, x_3 \dots x_n) + (y_1, y_2, y_3, \dots y_n)$$
$$= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$

and

$$kX = k(x_1, x_2, \dots, x_n)$$
$$kX = (kx_1, kx_2, kx_3, \dots, kx_n)$$

Is vector spase.

Sol.

let $x, y, z \in \mathbb{R}^n$, $k, m \in \mathbb{R}$ اي عنصرين

$x = (x_1, x_2, \dots, x_n)$ $Z = (z_1, z_2, \dots z_n)$ two Scal.

$y = (y_1, y_2, \dots y_n)$

① $x + y = (x_1, \dots, x_n) + (y_1, y_2, \dots y_n)$
$= ((x_1 + y_1), \dots, (x_n + y_n)) \in \mathbb{R}^n$

② $x + y = (x_1 + y_1 \dots x_n + y_n)$
$= (y_1 + x_1 \dots y_n + x_n)$
$= (y_1, \dots, y_n) + (x_1, \dots, x_n)$
$= y + x$

③ $x + [y + z] = x_1, x_2, \dots x_n + [(y_1, y_2, \dots, y_n) + (z_1, z_2, \dots z_n)]$
$= (x_1, x_2 \dots x_n) + ((y_1 z_1) + y_2 z_2, \dots y_n z_n)$
$= (x_1 + y_1 + z_1 \dots x_n + y_n + z_n)$
$= (x_1 + y_1 \dots x_n + y_n)(z_1 + \dots z_n)$
$= [x + y] + z$

④ $(0, 0, 0, \dots 0) \in \mathbb{R}^n$
let $0 = (0, 0, \dots 0)$

$X + 0 = (x_1, \dots x_n) + (0, \dots 0)$
$= (x_1, \dots x_n)$
$= x$
$= 0 + x$

⑤ If $X = (x_1, \dots x_n) \in \mathbb{R}^n$
take $(-x) = (-x_1, \dots, -x_n)$
$x + (-x) = (x_1, \dots x_n) + (-x_1, \dots -x_n)$
$= (x_1 - x_1, \dots x_n - x_n)$
$= 0$
$= (-x) + x$

Example (3):- $(M, +, \cdot)$, $M_{m \times n}$ is the set of all matrices sates of size $m \times n$, with usuall addition and scalar multiplication.

This is avector space. For the proof see Theorem (1,4,1).

The zero vector it the zero matrix of size $m \times n$. 0

Example (4):

$(V, +, \cdot)$ where V is the set of all real-valued Functions on $\mathbb{R}$ $(F_1 | \mathbb{R} \to \mathbb{R})$

and

$(f+g)(x) = f(x) + g(x)$ تعريف

$(kf)(x) = k(f(x))$ These is a vector space, For the proof, It is clearly that (1, 2, 3, 7, 8, 9, 10) are hold.

(4) The zero vectors is The zero function

$O : \mathbb{R} \to \mathbb{R} \ni O(x) = 0 = 0 \quad \forall x \in \mathbb{R}$

$(f+0)(x) = f(x) + 0(x)$ لازم تعرف

$= f(x) + 0$ ال زيرو مش

$= f(x)$ صفر اختص

$f + 0 = f$

(5) If $f \in V$, $-f \in V$

where $(-f(x)) = -f(x)$

$(f + -f)(x) = f(x) - f(x)$

$= f + (-f) = 0$

(6) $K \in \mathbb{R}$, $f \in V$, $(Kf) \in V$

since $(Kf)(x) = Kf(x) \in \mathbb{R} \quad \forall x$

$\Rightarrow Kf \in V$

Example (5)

$(\{0\}, +, \cdot)$ The zero vector space.

where

$0 + 0 = 0$

$k0 = 0 \quad \forall k \in \mathbb{R}$

Ex (6) plane through the origin

$(V, +, \cdot)$ where

$V = \{(a, b, c) \in \mathbb{R}^3$

$ax + by + cz = 0\}$

with Usual operation on $\mathbb{R}^3$. العمليات المعتادة

let: $(a, b, c) + (\grave{a}, \grave{b}, \grave{c}) = (a + \grave{a}, b + \grave{b}, c + \grave{c})$

$k(a, b, c) = (ka, kb, kc)$

$A = (a, b, c)$

$B = (\acute{a}, \acute{b}, \acute{c})$

$C = (\grave{a}, \grave{b}, \grave{c})$

$A \oplus B$

(1) $(a, b, c) + (\acute{a}, \acute{b}, \acute{c}) \in V$??

$(a + \acute{a}, b + \grave{b}, c + \grave{c}) \in V$??

$(a + \grave{a})x + (b + \grave{b})y + (c + \grave{c})z =$

$= (ax + by + cz) + (\grave{a}x + \grave{b}y + \grave{c}z)$

$0 + 0 = 0$

(2+3) hold.

(4) $0 = (0, 0, 0) \in V$??

$0(x) + (0)y + (0)z = 0$ ✓

⑤ $(a,b,c) \in V \Rightarrow (-a,-b,-c) \in V$??

$ax + by + cz = 0$

$(-a)x + (-b)y + (-c)z = 0$

⑥ $(a,b,c) \in V \Rightarrow k(a,b,c) \in V$??

$ax + by + cz = 0$

$(ka)x + (kb)y + (kc)z = 0$

(7) + (8) + (10) are hold

$R \to \{a\}$ example

$R^2 \to \{a,b\}$

$R^3 \to \{a,b,c\}$

---

every plane through the origin is a vector space

if $(x_1, y_1, z_1) \in V$

(x_2, y_2, z_2)

$ax_1 + by_1 + cz_1 = 1$

$ax_2 + by_2 + cz_2 = 1$

but $(x_1+x_2, y_1+y_2, z_1+z_2) \notin V$

since $a(x_1+x_2) + b(y_1+y_2) + c(z_1+z_2)$

$= a(x_1 + by_1 + cz_1) + a(x_2 + by_2 + cz_2)$

$= 1 + 1 = 2 \neq 1$

Example (7)

$(N, +, \cdot)$ usuall operation

① $0 \notin N$

2) $-n \notin N \quad \forall n$

3) $kn \notin N$, $k = \sqrt{2}$

Not vector space.

Example

⑧ $(R^2, +, \cdot)$

الشرط لما رجع لي صحيح

التعريف $\longrightarrow k(x,y) = (kx, 0)$

$(a,b) + (x,y) = (a+x, b+y)$

$k(a,b) = (ka, 0)$ ← الشرط

الشرط b لما شو الذي يحقق

$1(a,b) = (a,0) \neq (a,b) \quad b \neq 0$

Not a vector space.

Example (9)

$(V, +, \cdot)$ where $V = \{(a,b,c) \in R^2 : ax+by+cz=1\}$

$ax + by + cz = 1$ with usuall operation on R^3

If (a,b,c), $(a',b',c') \in V$

$ax + by + cz = 1$ and

$a'x + b'y + c'z = 1$

$(a+a')x + (b+b')y + (c+c')z = 2 \neq 1$

we all mostly

$(a+\acute{a}, b+\acute{b}, c+\acute{c}) \notin V$

$(0,0,0) \notin V$

$0x + 0y + 0z = 0 \neq 1$

Not vector space. لأنه لا يتحقق الأول

Example(10) $(V,+,\cdot)$ where

$$V = \left\{ \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}, a, b \in \mathbb{R} \right\}$$

with usuall operation.

Sol.

If $k=2$ the

$$k\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} = 2\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} = \begin{bmatrix} 2a & 2 \\ 2 & 2b \end{bmatrix} \notin V$$

Not avector space

---

problem 13 Homework

problem 15 Page (227)

(d) $kU = 0 \Rightarrow k = 0, u = 0$

assume $ku = 0$

if $k = 0$ then it is done

if $k \neq 0$ then $\frac{1}{k} \in \mathbb{R}$

$\frac{1}{k} \cdot ku = \frac{1}{k} 0$

$1 \cdot u = 0 \Rightarrow u = 0$

Theorem (5,1,1)

let V be avector space let $u \in V$ and k is any scalar, then

a) $0u = 0$

b) $k0 = 0$

c) $(-1)u = -u$

d) If $ku = 0$, then $k = 0$ or $u = 0$

proof

a) $0u + 0u = (0+0)u = 0u$ adding $-0u$

$-0u + 0u + 0u = -0u + 0u$

$0 + 0u = 0$

$\boxed{0u = 0}$ #

b) $k0 + k0 = k(0+0) = k(0)$

$-k0$ as a) $k0 = 0$ $-k0$

c) $u + (-1)u = 1 \cdot u + (-1)u = (1+(-1))u = 0u = 0$ add $(-u)$

$-u + u + (-1)u = 0 + (-u)$

$\boxed{(-1)u = -u}$ #

Proof Excersise

(d)

we support the ~~th~~ EL Zamalek club by our hearts before

The End of the section 5.1

أهلاً يا شعب

all Muslims

الفضاءات الجزئية

section 5.2 Subspaces

def:- A subset W of a vector space V is called a subspace of V is called a subspace of V if W is it self a vector space under the **same** operation defined **on** V.

شرط اساسي

Theorem (5.2.1):- If W is a non empty subset of a vector space V, then W is a subspace of V iff the following hold:-

الأول a) If u and $v \in W$, then $u+v \in W$
(Closed under addition) مغلق على الجمع

الثاني b) If $u \in W$, $k \in R$, then $ku \in W$
(Closed under scalar multiplication)

** اثبات مش مطلوب .

Subspaces
- $W \neq \phi$
- closed add. $u+v \in W$
- closed multiple. $ku \in W$; $k \in R$, $u \in W$

كتابة

Ex (1) أثبت أثبتيا

$(V, +, \cdot) \subseteq (R^3, +, \cdot)$

Where

$V = \{(a, b, c) \in R^3 \; ax + by + cz = 0\}$

is a subspace.

Example (2) أثبت

$(P_n(x), +, \cdot) \subseteq (F(R), +, \cdot)$

where

$P_n(x) = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \; , \; a_i \in R\}$

$F(R) : [f : R \to R]$

$(f+g)(x) = f(x) + g(x)$

$(kf)(x) = k \cdot f(x)$

Sol. $P_n(x) \neq \phi$ since $0 \in P_n(x)$

a) let $f, g \in P_n(x)$

$f(x) = a_0 +$

$g(x) =$

$(f+g)(x) =$

$kf(x)$

$=$

So $(P_n(x), +, \cdot)$ is a subspace of $(F(R), +, \cdot)$ #

---

(3) $(S, +, \cdot) \subseteq (M_{n \times n}, +, \cdot)$

$S = \{A \in M_{n \times n} : A \text{ is symmetric}\}$

$= \{A \in M_{n \times n} : A^T = A\}$ ← sym

clearly $S \neq \phi$: $(0) I_n \in S$

If $A, B \in S \Rightarrow A^T = A \quad B^T = B$

$(A+B)^T = A^T + B^T = A + B \Rightarrow A + B \in S$

$(kA)^T = kA^T = kA \Rightarrow kA \in S$

sub space

ال matrix

not sup space ← inversable

sup space ← symmetric

Example (4)

$(V, +, \cdot) \subseteq (\mathbb{R}^3, +, \cdot)$ where

$V = \{(a, b, 0) : a, b \in \mathbb{R}^3$ with

usual operation.

$(a, b, 0) \in \mathbb{R}^3$

$V \neq \phi$ since $(0,0,0) \in V$

① let $(a,b,0), (c,d,0) \in V$

$(a,b,0) + (c,d,0) = (a+c, b+d, 0)$

② If $k \in \mathbb{R}$, $(a,b,0) \in V$

$k(a,b,0) = (ka, kb, 0)$

So it is a supspace.

Ex (5) :- $(V, +, \cdot) \subseteq (\mathbb{R}^3, +, \cdot)$.

$V = [(a, b, 1) : a, b \in \mathbb{R}$

Not a subspace, since $(a, b, 1) + (c, d, 1)$

$= (a+b), b+c, 2) \notin V$

Example (6) $(V, +, \cdot) \subseteq (\mathbb{R}^2, +, \cdot)$ where

$V = \{(x, y) : x \geq 0, y \geq 0\}$

Sol.

$V \neq \phi$ since $(0, 0) \in V$.

1) $(x, y), (a, b) \in V$

$x, y, a, b \geq 0$

$(x, y) + (a, b) = (x+a, y+b) \in V$

since $x + a \geq 0$

$y + b \geq 0$

3) If $k \in \mathbb{R}$ $(x, y) \in V$ $k \cdot (x, y) = (kx, ky) \notin V$

since if $k < 0$, the

$kx \leq 0$, $ky \leq 0$, since

$(1, 1) \in V$ $-1, \in \mathbb{R}$

but $-1(1,1) = (-1,-1) \notin V$ since
$-1 < 0$
$-1 < 0$
الشرط الأول
So it is Not a subsequen

Theorem:- (5.2.2)

If $AX = 0$ a homogenous linear sys
homogenou
of (m) equations (n) ~~unknowns~~ then
unknowned
عدد المجاهيل (n)
the set of solution vectors is a subspace of $\mathbb{R}^n$

Proof:-

let V = The set of all solution of $AX = 0$

$\Rightarrow V = \{ x \in \mathbb{R}^n : AX = 0 \}$

$= \{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : AX = 0 \}$

Clearly, $0 \in V$ $(0 = (0, 0, 0, \ldots, 0))$
since $AX = 0$ has the trivial Solution

let $x_1, x_2 \in V$

$AX_1 = 0$, $AX_2 = 0$
$A(X_1 + X_2) = AX_1 + AX_2 = 0 + 0 = 0$
$\Rightarrow X_1 + X_2 \in V$

If $k \in \mathbb{R}$, $X_1 \in V$
$A(k)X_1 = k(AX_1) = k(0) = 0$
$kX_1 \in V$

$\Rightarrow$ V subspace

Exercise

مجاهيل

$$\begin{bmatrix} 2 & -1 & 0 \\ 4 & 3 & 1 \\ -2 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$V = \{(0, 0, 0)\}$

because A is invertible then is trivialsubspace.

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 4 & 7 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$V = \{ (x, y, z) = (-5t, 3t, t) \; ; \; t \in \mathbb{R}$

c) $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ 2 & 6 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

الحل $V = \{(x,y,z) : x+2y-z=0\}$ plane through the origin normal the vector $i+2j-k$

d) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

الحل $V = \mathbb{R}^3$ the entire space

def:- A vector w is called a <u>Linear Combination</u> (تركيبة خطية) of the vectors $V_1, V_2, V_3, \ldots, V_r$

If we can write $w = k_1V_1 + k_2V_2 + \cdots + k_rV_r$ where $k_i \in \mathbb{R}$ $\forall i \leq r$

Example: ① Every element in $\mathbb{R}^3$ is a linear combination of the vectors $i = (1,0,0)$, $j = (0,1,0)$, $k = (0,0,1)$. since $(a,b,c) = ai + bj + ck$

Ex 9 (الكتاب)

② Show that $w_1 = (9,2,7)$ is a linear combination and $w_2 = (4,-1,8)$ is not a linear combination of $u = (1,2,-1)$ and $v = (6,4,2)$ in $\mathbb{R}^3$

Sol:

we will search for a constants $k_1, k_2 \ni$

$$w_1 = k_1u + k_2v$$

$(9,2,7) = k_1(1,2,-1) + k_2(6,4,2)$

$(9,2,7) = (k_1+6k_2,\ 2k_1+4k_2,\ -k_1+2k_2)$

$9 = k_1 + 6k_2$

$2 = 2k_1 + 4k_2$

$7 = -k_1 + 2k_2$ we will solve this system

$$\begin{bmatrix} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{bmatrix} \xrightarrow{ERO} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

then

$k_1 = -3$, $k_2 = 2$

$w_1 = -3u + 2v$ L.C.

So, w is a Linear Compi (L.C) of u and v

وأنت لحالك حل w_2

(b) by same method we will obtain

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 8 \end{bmatrix} \xrightarrow{ERO} \begin{bmatrix} 1 & 0 & 4 \\ 0 & -2 & -9 \\ 0 & 0 & 3 \end{bmatrix}$$

this system is with no solution so w_1 is not a Linear Combination (L.C) of of u and v.

Theorem 5.2.3

If $v_1, v_2 \ldots v_r$ are vectors in a vector space V, then

a) The set W of all linear combinations of $v_1, v_2, \ldots v_r$ form subspace of the space V.

b) W is the smallest ~~subspace~~ supspace of V that contain $v_1, v_2, \ldots v_r$
That is, If T is another space contain $v_1, v_2, \ldots v_r$ then $W \subset T$:-

Proof:-

a) $W = \{a_1v_1 + a_2v_2 + \cdots + a_rv_r : a_i \in R\}$
since $0v_1 + 0v_2 + \cdots + 0v_r = 0$, then $0 \in W$ and $W \neq \phi$

let $a_1v_1 + a_2v_2 + \cdots + a_rv_r \in W$
and $b_1v_1 + b_2v_2 + \cdots + b_rv_r \in W$

$(a_1v_1 + a_2v_2 + \cdots + a_nv_n) + (b_1v_1 + b_2v_2 + \cdots + b_rv_r)$

$= ((a_1+b_1)v_1 + (a_2+b_2)v_2 + \cdots + (a_r+b_r)v_r) \in W$

If $k \in R$

$k(a_1v_1 + \cdots + a_rv_r) = (ka_1)v_1 + \cdots + (ka_r)v_r \in W$

$\boxed{W \text{ is a subspace of } V}$

b) let T be another subspace $\ni v_1, v_2, \ldots v_r \in T$.

let $a_1v_1 + a_2v_2 + \cdots + a_rv_r \in W$
then since T is a subspace, hence T is closed under addition and scalar multiplication, then $a_1v_1 + \cdots + a_rv_r \in T$
& $W \subseteq T$.

الطلاب بيستعملوا منه spanned تكلم الأستاذ ورقة

التوليد والتمثيل ص 236

Def:- spanned

If $S = \{v_1, v_2, \dots, v_r\}$ is a set of element in a vector space V, then the subspace W of V consisting of all linear combination of the vectors is called the subspace spand by $v_1, v_2, \dots, v_r$ and we say the vectors $v_1, v_2, \dots, v_r$ ~~and we~~ spand W. we write $W = \text{span}\{v_1, v_2, \dots, v_r\}$

$$\boxed{W = \text{Span } S}$$

span تسمى مجموعة العناصر على شكل Linear combination مجموعة من

Example: [1] let $V = R^3$. let $(x,y,z) \in R^3$

let $W = \text{span}\{(x,y,z)\}$

$= \{a(x,y,z) : a \in R\}$

$= \{(ax, ay, az) : a \in R\}$

is a line pass through the origin.

تكون span اذا كان هناك حل للنظام

يعني invertible $\det A \neq 0$

$(b_1, b_2, b_3) = k_1v_1 + k_2v_2 + k_3v_3$

يعني تكون Linear combination

[2] let $V = R^3$, $a, b \in R^3$. $a \neq kb$ $\forall k \in R$

$a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$

$W = \text{span}(a, b)$

$= (k_1(a_1, a_2, a_3) + k_2(b_1, b_2, b_3))$; $k_1, k_2 \in R$

$= \{(k_1a_1 + k_2b_1, k_1a_2 + k_2b_2, k_1a_3 + k_2b_3)\}$ $k_1, k_2 \in R$

is a plane through the origin.

(3) $V = R^3$, $i = (1, 0, 0)$, $j = (0, 1, 0)$, $k = (0, 0, 1)$

then ~~we~~ $W = R^3$

$R^3 = \text{span}\{i, j, k\}$

(4) let $V = P_n(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n\}$; $a_i \in R$

then $P_n(x) = \text{Span}\{1, x, x^2, \dots, x^n\}$

since $\underset{v_1}{a_0} + \underset{v_2}{a_1x} + \underset{v_3}{\dots} + a_nx^n \in P_n(x)$

$a_0(1) + a_1(x) + a_2(x^2) + \dots + a_nx^n$

Example: Determine if $V_1 = (1,1,2)$
$V_2 = (1,0,1)$ $V_3 = (2,1,3)$ span $\mathbb{R}^3$

Sol. let $(b_1, b_2, b_3) \in \mathbb{R}^3$ we must search for constan $k_1, k_2, k_3 \ni (b_1, b_2, b_3) = k_1 V_1 + k_2 V_2 + k_3 V_3$

$$b_1 = k_1 + k_2 + 2k_3$$
$$b_2 = k_1 + k_3$$
$$b_3 = 2k_1 + k_2 + 3k_3$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 1 & 0 & 1 & b_2 \\ 2 & 1 & 3 & b_3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix}$$

This system has a solution iff $\Leftrightarrow$
$$b_3 - b_2 - b_1 = 0$$
$$b_3 = b_2 + b_1$$
Now, If we take any element with this condition doesn't hold, then we cannot find k_1, k_2, k_3 $(0,0,1) \in \mathbb{R}^3$, and
$(0,0,1) \neq k_1 V_1 + k_2 V_2 + k_3 V_3$ for any $k_1, k_2, k_3 \in \mathbb{R}$ ~~of~~ has mean
V_1, V_2, V_3 Cannot span $\mathbb{R}^3$.

Theorem (5.2.4)
If $S = (v_1, v_2, \dots v_r)$ and $\bar{S} = (w_1, w_2, \dots w_k)$ are two sets of vectors in a vector space V then span $(v_1, v_2, \dots v_r)$ = span $(w_1, w_2, \dots w_k)$ iff each vectors in S is a linear combination of those in $\bar{S}$ and each vectors in $\bar{S}$ is a linear combinations of those in S

Proof. Exercise - ~~تمرين ٢~~ تمرين

الاستقلال الخطي

Section (5.3) Linear ~~Combination~~ Independent :- تعامل حل النظام

def: If $S=\{V_1,V_2,\dots,V_r\}$ is a non empty set of vector then the vector equation

$k_1V_1+k_2V_2+\dots+k_rV_r=0$ has at least one solution namely $k_1=0, k_2=0, \dots, k_r=0$
حل صفري

If this in the only solution, then S is called a linear independent set.
خطية مستقلة

If there are other solution then S is called a linear dependent set.
حلول أخرى

اذا كان له حل صفري ← independent

اذا كان له حل غير صفري ← dependent

Example: ① $V_1=(2,-1,0,3)$, $V_2=(1,2,5,-1)$
$V_3=(7,-1,5,8)$

det A = 0 LD الحل اي

LI حل صفري

$S=\{V_1,V_2,V_3\}$

$3V_1+V_2-V_3=3(2,-1,0,3)+(1,2,5,-1)-(7,-1,5,8)$
$=(0,0,0,0)=0$

$k_1V_1+k_2V_2+k_3V_3=0$
$\Rightarrow k_1=3, k_2=1, k_3=-1$

So, S is Linear dependent set.

② $P_1=1-x$, $P_2=5+3x-2x^2$; $P_3=1+3x-x^2$

$S=\{P_1,P_2,P_3\}$

$3P_1-P_2+2P_3=0$

$k_1P_1+k_2P_2+k_3P_3=0$

$k_1=3$
$k_2=-1$
$k_3=2$

So Linear dependent.

---

③ $S=\{i,j,k\}$

let

$k_1 i+k_2 j+k_3 k=0$

$(k_1,0,0)+(0,k_2,0)+(0,0,k_3)=(0,0,0)$

$k_1=0$, $k_2=0$, $k_3=0$ ← only one solution زيرو

So is a linear Independent.

---

④ $[(1,-2,3), (5,6,-1), (3,2,1)]$
V_1, V_2, V_3

الحل الصفري

$k_1V_1+k_2V_2+k_3V_3=0$ (X) غلط

$k_1(1,-2,3)+k_2(5,6,-1)+k_3(3,2,1)=0$

$k_1-2k_1+3k_1+5k_2+6k_2-k_2+3k_3+2k_3+1k_3=(0,0,0)$

$2k_1+10k_2+6k_3=(0,0,0)$

$k_1+5k_2+3k_3=0$ ←
$-2k_1+k_2+2k_3=0$ ←
$3k_1+k_2+k_3=0$ ←

$$\left[\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ -2 & 6 & 2 & 0 \\ 3 & 1 & 1 & 0 \end{array}\right] \xrightarrow{ERO} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\boxed{k_1 = 0}$ $\boxed{k_2 = 0}$ $\boxed{k_3 = 0}$ then L.I

---

(5) $\{1, x, x^2, \ldots x^n\}$ are Linear Independent is f_n

$k_1(1) + k_2(x) + k_3(x^2) + \ldots k_{n+1} x^n = 0$

$\boxed{x = 0} \Rightarrow k_1 = 0$

after diff:-

$k_2 + 2k_3 x + \ldots + n k_{n+1} x^{n-1} = 0$

$\boxed{x = 0}$ $\boxed{k_2 = 0}$

$\boxed{x = 1}$ $n! \, k_{n+1} = 0$

$\boxed{k_{n+1} = 0}$

$k_1 = k_2 = k_3 = 0 = k_{n+1}$

$\Rightarrow$ L.I #

(Theorem 5.3.1)

A set S with two or more vectors is :-

(a) إذا انفطعت ___ أو أثبت عنصر واحد على الأقل على شكل Linea Combin

a) linearly dependent iff at least one of these vectors is expresible as a linear Combination of the other vectors in S

(b) Linearly independent iff no vector in S is exprisible as a linear Combination of the other vectors in S

Proof :- take $S = \{v_1, v_2, \ldots v_r\}$, $r \geq 2$

a) Assum that S is linearly dependent that mean the vector equation $k_1v_1 + k_2v_2 + \ldots + k_rv_r = 0$ has non trivial Solution. this mean at least one scalar is not equal zero.

let this scalar be k_1

$k_1v_1 + k_2v_2 + \ldots + k_rv_r = 0$

and $k_1 \neq 0$

$$v_1 = -\left[\frac{k_2}{k_1}v_2 + \frac{k_3}{k_1}v_3 + \ldots + \frac{k_r}{k_1}v_r\right]$$

$\Rightarrow$ So v_1 is linear Combination of the other vectors in S.

$\Leftarrow$ assume at least one of these vectors is a linear Combination of the other vectors.

let this vector (after rearragment) be V_1

$$V_1 = C_2V_2 + C_3V_3 + \cdots + C_rV_r$$

$$(1)V_1 + (-C_2)V_2 + (-C_3)V_3 + \cdots + (-C_r)V_r = 0$$

$\boxed{k_1 = 1}$, $k_2 = -C_2$, ---
has a non trivial solution
So, S is linearly dependent set.

proof (b) the contrapositive of (a)

Example: 1) $\{V_1, V_2, V_3\}$ in Ex (1)

2) $\{i, j, k\}$ in Enercise (3)

(a see Ex 1 $\quad v_3 = 3v_1 + v_2 \quad v_2 = v_3 - 3v_1 \quad v_1 = [v_3 - v_2]/3$

(b see Ex 3

Theorem (5.3.2)

a) A finite set vectors that contains the zero vectors is linearly Dependent { اذا مجموعة فيها vector zero هي linear dep على طول }

b) A set with exactly two vectors is linearly independent iff neither vector is scalar multiple of the other vector
المجموعة اللي فيها جزئين باحكي عنها linear indep اذا ما كانش واحد Multiple من الثاني

Proof (a) let $S = \{0, V_1, V_2, V_3, \ldots, V_r\}$. let

$$k0 + k_1V_1 + k_2V_2 + \cdots + k_rV_r = 0$$

this vector ~~sp~~ equation has nontrivial Sol:
$\boxed{k = 1}$, $k_1 = k_2 = \cdots = k_r = 0$
So, S is linearly Dependent. #

(b) let $S = \{V_1, V_2\}$ is L.I (linearly independent) set iff we cannot write one of these vectors, as a linear Combination of the other iff
$$V_1 \neq kV_2 \quad \forall k \in \mathbb{R}$$
iff no one of them is a scalar multiple of ~~the~~ other #

تابع ورقة

Example: [1] $\{x, \sin x\}$ is L.I set since $x \neq k \sin x \ \forall k \in R$

[2] $\{0, 2x, x^3\}$ is linearly dependent set since $0 \in \{0, 2x, x^3\}$

[3] $\{x, \frac{x}{2}\}$ is linearly Depende set since $x = 2\left(\frac{x}{2}\right), 2 \in R$

Theorem (5.3.3) let $S = \{v_1, v_2, \ldots, v_r\}$ be a set of vectors is R^n. If $r > n$, then S is L.D set.

r — عدد المتجهات =

n — عدد العناصر المتجه الواحد

Example:

a) $\{(1,2), (3,4), (5,6)\}$ Linearly Dependent.

R^2 3 vector $\xrightarrow{\text{then}}$ L.D.

والعكس ليس صحيح ?? واضح

b) $\{(1,1,1), (2,2,2)\}$ L.D.

واحدة Multiple من الثانية ??

Linear Independence of Function:-

Def.- let $f_1 = f_1(x), f_2 = f_2(x), \ldots f_n = f_n(x)$ are $n-1$ times differentiable functions on R. Then we define the Wronskin of $f_1, f_2, \ldots, f_n$ denoted by $W(f_1, f_2, \ldots f_n)$ as

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & f_3'(x) & \cdots & f_n'(x) \\ \vdots & & & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & & f_n^{(n-1)}(x) \end{vmatrix}$$

Note that $W(x)$ is function on R.

Theorem (5.3.4) $(-\infty, \infty)$

If the functions $f_1, f_2, \ldots, f_n$ have $n-1$ continous derivatives on the interval $(-\infty, \infty)$ and if the wronskin of these functions is not identitically zero on $(-\infty, \infty)$ then these functions form a linear ~~identity~~ independent set if vector in $C^{n-1}(-\infty, \infty)$ [The vector space of all $(n-1)$ time diffi functions on R with usual operation]

Example: $\{x, \sin x, \cos x\}$

بيعطيني كلهم الدوال مالة
اشتقاق الى الازم N.

$$W(x) = \begin{vmatrix} x & \sin x & \cos x \\ 1 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{vmatrix}$$

$$= x\left[-\cos^2 x - \sin^2 x\right] - \cancel{[\sin x(\cos x) + \cos x]} - 1\left[-\cos x \cancel{\sin x} + \cos x \sin x\right]$$

$$= x[-1] = -x \qquad \boxed{= -x}$$

$W(x) = -x$

$W(1) = -1 \neq 0$

So, $\{x, \sin x, \cos x\}$ is Linearly independent set

b) $\{x, e^x, e^{2x}\}$

$$W(x) = \begin{vmatrix} x & e^x & e^{2x} \\ 1 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{vmatrix}$$

$$= x\left[4e^{3x} - 2e^{3x}\right] - \left[4e^{3x} - e^{3x}\right]$$

$$= x\,2e^{3x} - 3e^{3x}$$

$$= e^{3x}[2x - 3]$$

$W(0) = -3 \neq 0$

So, $\{x, e^x, e^{2x}\}$ is Linear Independent

check

c) $\{x, 2x, x^3\}$ L.I or LD

$$W(x) = \begin{vmatrix} x & 2x & x^2 \\ 1 & 2 & 2x \\ 0 & 0 & 2 \end{vmatrix}$$

$$= 2[2x - 2x] = 0$$

$W(x) = 0 \quad \forall x \in R$

this method doesn't give us any information in this case, so we try another method

Clearly $\{x, 2x, x^2\}$ is L.D

Remark:- The Converse of last theorem is not true.

Example:- $\{x^2, x|x|\}$

$$W(x) = \begin{vmatrix} x^2 & x|x| \\ 2x & 2|x| \end{vmatrix} = 0 \quad \forall x \in R$$

but, $\{x^2, x|x|\}$ is L.I set

$ax^2 + bx|x| = 0 \quad \forall x \in R$

$a = b = 0$

Section (4) Basis and Dimension :-

Def.:- If V is any vector space and $S = \{v_1, v_2, \dots v_r\}$ is a set of vectors in V, then S is called a basis for V if the following two conditions hold :-

a) S is Linearly Independen s.

b) S spans V.

Example: ① $\{i, j, k\}$ is a basic for R^3.

In general the set

$[(1,0,0,\dots 0), (0,1,0,\dots 0), \dots (0,0,0,\dots,1)]$

is basis for R^n.

② $\{1, x, x^2, \dots, x^n\}$ is a basis for $P_n(x)$.

③ $\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$

is a basis for $M_{2\times 2}$

④ If $S = \{v_1, v_2, \dots v_r\}$ are linearly independent set in V

the S is basis for the subspace generated by the vectors in S.

مهم ⑤. Show that فش (امتحان) يخلو من سؤال مثل هذا

$v_1 = (2,0,-1)$, $v_2 = (4,0,7)$, $v_3 = (-1,1,4)$ ~~for~~ a basis for R^3.

① is L.I ? راجع لنك طلعة [illegible]

take

$k_1v_1 + k_2v_2 + k_3v_3 = 0$

$k_1(2,0,-1) + k_2(4,0,7) + k_3(-1,1,4) = 0$

$2k_1 + 4k_2 - k_3 = 0$

$k_3 = 0$

$-k_1 + 7k_2 + 4k_3 = 0$

$\boxed{k_1 = k_2 = k_3 = 0}$ [illegible]

So, $\{v_1, v_2, v_3\}$ is linearly independent set.

② Span ? let $(x,y,z) \in R^3$. we must find constant $a_1, a_2, a_3 \ni$

$a_1v_1 + a_2v_2 + a_3v_3 = (x,y,z)$

$\{$

$2a_1 + 4a_2 - a_3 = x$

$a_3 = y$

$-a_1 + 7a_2 + 4a_3 = z$

$$\begin{bmatrix} 2 & 4 & -1 \\ 0 & 0 & 1 \\ -1 & 7 & 4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & -1 & x \\ 0 & 0 & 1 & y \\ -1 & 7 & 4 & z \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 4 & -1 & x \\ -1 & 7 & 4 & z \\ 0 & 0 & 1 & y \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -\frac{1}{2} & \frac{x}{2} \\ 0 & 9 & -\frac{1}{2} & z - 4y - \frac{x}{2} \\ 0 & 0 & 1 & y \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & \frac{x}{2} + \frac{y}{2} \\ 0 & 1 & 0 & \frac{1}{9}(z + \frac{7}{2}y - \frac{x}{2}) \\ 0 & 0 & 1 & y \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{x}{2} + \frac{y}{2} - \frac{2}{9}(z - \frac{7}{2}y - \frac{x}{2}) \\ 0 & 1 & 0 & \frac{1}{9}(z + \frac{7}{2}y - \frac{x}{2}) \\ 0 & 0 & 1 & y \end{bmatrix}$$

$$\boxed{a_1 = \frac{11}{18}x + \frac{23}{18}y - \frac{2}{9}z \qquad a_2 = \frac{z}{9} - \frac{7}{18}y - \frac{x}{18}} \qquad \boxed{a_3 = y}$$

we find the scalar to solve the equation and that mean (x, y, z) is a linear combination of v_1, v_2, v_3 and this implies $\{v_1, v_2, v_3\}$ span R^3.

So $\{v_1, v_2, v_3\}$ is a basis for R^3.

#

Example (6):- $\{(2,-1,0,3), (1,2,5,-1), (7,-1,5,8), (2,3,4,1)\}$

not a basis for R^4

Since these vectors ~~are~~ form a linear dependent set

حسب ما قيل هيك لأن عناصر S depn ولو اضفنا لهم أي شي بتبقى dep.

Ex(7) $\{(1,1,2), (1,0,1), (2,1,3)\}$ not a basis for R^3 since the cannot span R^3.

~~not span~~

not basis bean not span.

Theorem:- (5.4.1)

If $S = \{v_1, v_2, \dots, v_n\}$ is a basis for a vector space V, then every vector $v \in V$ can be expressed in the form

$v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$

in exactly one way.

لو كان عندك قاعدة لفضاء معين بالتالي أي عنصر فيه

Proof:- ممكن كتابته على شكل تركيب خطي من عناصر القاعدة

بهذا الشكل بشكل وحيد

لأن

Proof: The exsistance of this form comes from that S is a basis for V.

Now, let $v \in V$, let

$$u = c_1v_1 + c_2v_2 + \cdots + c_nv_n$$
&
$$v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$$

$$0 = (v - v)$$
$$= (c_1v_1 + c_2v_2 + \cdots + c_nv_n) - (a_1v_1 + a_2v_2 + \cdots + a_nv_n)$$
$$= (c_1 - a_1)v_1 + (c_2 - a_2)v_2 + \cdots + (c_n - a_n)v_n$$

since $\{v_1, \ldots, v_n\}$ are L. I set then

$$c_1 - a_1 = 0,\ c_2 - a_2 = 0,\ c_n - a_n = 0$$
$$c_1 = a_1,\ c_2 = a_2 \cdots c_n = a_n$$

So this form is a unique.

Def. If $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for a vector space V and $v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$ then the scalars $c_1, c_2, \ldots, c_n$ are called <u>the coordinates</u> (محاوات) of V relative (نسبة) to a basis S.
The vector $c_1, c_2, \ldots, c_n$ in R^n constructed from these coordinates is called the <u>coordinate vector</u> v relative to S, we <u>write</u>

$$v = c\, s$$

$$(v)_s = (c_1, c_2, c_3, \ldots, c_n)$$

Example: أسئلة تأتي في الامتحان مهمة جداً

If $v = (x_1, x_2, \ldots, x_n) \in R^n$ and $S = \{e_1, e_2, \ldots, e_n\}$ is the standard basis for R^n, where

$e_1 = (1, 0, 0, \ldots), e_2 = (0, 1, 0, 0, \ldots), \ldots, e_n = (0, 0, \ldots, 1)$

then $v = (x_1, x_2, \ldots, x_n)$

$v = c_1e_1 + c_2e_2 + \cdots + c_ne_n$

R^n فقط s standard bas

$(x_1, x_2, x_3 \ldots x_n) = c_1(1, 0, 0, \ldots) + c_2(0, 1, 0, 0 \ldots), \ldots + c_n(0, 0, \ldots, 1)$

$(x_1, x_2, \ldots, x_n) = (c_1, c_2, c_3, \ldots c_n)$
then

$c_1 = x_1,\ c_2 = x_2, \ldots, c_n = x_n$

In this case (only)

$(v)_s = v$

Note

S.b $R^n \longrightarrow S = \{e_1, e_2, \ldots e_n\}$

S.b $P_n(x) \longrightarrow S = \{1, x, x^2, \ldots, x^n\}$

هذه هي الحالة الوحيدة التي يكون فيها متجه الإحداثيات = المتجه نفسه
الوحيد متجه العنصر بأرقامه نفسه

② If $V = P_3(x)$ with standard basis

$S = \{1, x, x^2, x^3\}$ let

$f(x) = 2x^2 + 3x - 2$, To Find

$(f(x))_S$, we must solve

$$f(x) = c_1(1) + c_2(x) + c_3(x^2) + c_4(x^3)$$

~~$2x^2 + 3x - 2$ To Find $(f(x))_S$~~

$$-2 + 3x + 2x^2 = c_1 + c_2 x + c_3 x^2 + c_4 x^3$$

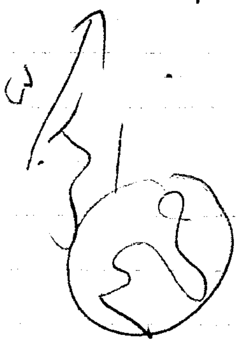
$c_1 = -2$, $c_2 = 3$, $c_3 = 2$, $c_4 = 0$

$$(f(x))_S = (-2, 3, 2, 0)$$

Example: let $v_1 = (2,4,-2)$, $v_2 = (-1,3,5)$
$v_3 = (0,1,6)$

a) Show that $S = \{v_1, v_2, v_3\}$ is a basis for R^3. (Ex يوم سبت)

سؤال



b) Find the coordinate vector of $v = (5,-1,9)$ relative to S.

Sol: we will solve

$$v = c_1 v_1 + c_2 v_2 + c_3 v_3$$

$$(5,-1,9) = c_1(2,4,-2) + c_2(-1,3,5) + c_3(0,1,6)$$

$$(5,-1,9) = 2c_1 - c_2 + 4c_1 + 3c_2 + c_3 + (-2c_1 + 5c_2 + 6c_3)$$

$$\begin{aligned} 2c_1 - c_2 &= 5 \\ 4c_1 + 3c_2 + c_3 &= -1 \\ -2c_1 + 5c_2 + 6c_3 &= 9 \end{aligned} \Longrightarrow \begin{bmatrix} 2 & -1 & 0 & 5 \\ 4 & 3 & 1 & -1 \\ -2 & 5 & 6 & 9 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 2 & -1 & 0 & 5 \\ 0 & 5 & 1 & -11 \\ 0 & 4 & 6 & 14 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -1 & 0 & 5 \\ 0 & 1 & -5 & -25 \\ 0 & 4 & 6 & 14 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 2 & -1 & 0 & 5 \\ 0 & 1 & -5 & -25 \\ 0 & 0 & 26 & 114 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 & 0 & 5 \\ 0 & 1 & -5 & -25 \\ 0 & 0 & 1 & \frac{114}{26} \end{bmatrix}$$

$c_3 = \frac{114}{26}$, $c_2 = -25 + \frac{570}{26}$, $c_1 = 5 - 25 + \frac{570}{26}$

$c_1 = \frac{114}{26}$, $c_2 = \frac{-80}{26}$, $c_3 = \frac{150}{26}$

$$(v)_S = \left(\frac{150}{26}, \frac{-80}{26}, \frac{114}{26}\right)$$

c) Find the vector V in R^3 whose coordinate vector with respect to S.
$(V)_S = (-1,3,2)$

$$V = c_1v_1 + c_2v_2 + c_3v_3$$

$$= -1(2,4,-2) + 3(-1,3,5) + 2(0,1,6)$$
$$= (-5, 7, 29)$$

محدود الابعاد

def. A non zero vector space V is called finite dimentional if it contains a finite (مجموعة محدودة) set of vectors $\{v_1, v_2, \dots, v_n\}$ that form a basis. اذا لقيت مجموعة محدودة من الفكتورز بتكون finit dimention

If no such set exsist, then V is called infinit dimentional غير محدود الابعاد ← اذا ما لقيتش شي

محدود الابعاد

The zero vector space is of finite dimension.

Ex: (1) R^n, $P_n(x)$, $M_{m\times n}$ are finite dimentional

(2) $F(R)$, $C(R)$, $C^n(R)$ are of infinite dimention

مجموعة الدوال المعرفة على R — مجموعة الدوال المتصلة — مجموعة الدوال القابلة للاشتقاق n من المرات

Theorem (5.4.2)

let V be a finite dimentional vector space let $v_1, v_2, \dots, v_n$ be any basis. then

(a) If a set has more than n elements, then this set is linearly dependent set.

(b) If a set has less than n elements then this set doesn't span V لا تولد V

$r > n$ LD

$\{v_1, v_2, \dots, v_n\}$

$n > r$ → Not Span

$\dim V = n$ $\boxed{n}$

$n < r$ → Linearly Dependent

Example: let $V = R^3$ then $\{i, j, k\}$ is a basis for V

$r = 4$

a) $S = \{(1,2,3), (0,1,5), (2,4,-2), (0,0,3)\}$ $n = 3$

This set is linearly Dependent. (how?)

$v_1, v_2, v_3, v_4 \leftarrow r$

$n = 3$, $r = 4$

$r > n$ then L.D.

b) $S = \{(1,2,3), (0,1,3)\}$ R^3 $n=3$, $r=2$, $n > r$

doesn't span R^3 since $(0,0,1)$ cannot written as linear combination of these element.

Theorem (5.4.3)
All basis for a finite dimensional vector space have the same number of elements.

V
$\downarrow$
$\{v_1, v_2, \dots, v_n\}$

$< n$: not span, not basis
$> n$: L.D, not basis

Def: The dimention of a finite dimentional vector space V denoted by $\dim V$ is the number of vectors in a basis of V. The zero vector space is of dimintion zero.

Example: ① $\dim R^n = n$ $S = \{e_1, e_2, \dots, e_n\}$
② $\dim P_n(x) = n+1$ $S = (1, x, x^2, x^3, \dots, x^n\}$
③ $\dim M_{m\times n} = m\times n$

$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & 0 \end{bmatrix}$ ------ $\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \cdots & & 1 \end{bmatrix}$

$\dim \{0\} = 0$

مثال $M_{2\times2}$: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$2\times2 = 4$

$AX = 0 \Rightarrow$ S.S is V.S

Example: Determined the basis and dimention for the solution space of the system.

$$\begin{aligned} 2x_1 + 2x_2 - x_3 + x_5 &= 0 \\ -x_1 - x_2 + 2x_3 - 3x_4 + x_5 &= 0 \\ x_1 + x_2 - 2x_3 - x_5 &= 0 \\ x_3 + x_4 + x_5 &= 0 \end{aligned}$$

by Exer 7 sec 1

Sol.
$x_2 = s$, $x_4 = 0$, $x_5 = t$

The solution of this system is
$x_1 = -s-t$ $x_3 = -t$
$x_2 = s$ $x_4 = 0$
$x_5 = t$

((Ex. 7) sec. 1.2)

$\dim \{0\} = 0$

$W = \{(x_1, x_2, x_3, x_4, x_5)\} = \{(-s-t, s, -t, 0, t)\}$

subspace

$= \{(-s, s, 0, 0, 0) + (-t, 0, -t, 0, t)$
$\quad s u_1 + t u_2$
$= \{s(-1, 1, 0, 0, 0) + t(-1, 0, -1, 0, 1)\}$

$= \text{span}\{(-1, 1, 0, 0, 0), (-1, 0, -1, 0, 1)\}$

clearly u_1, u_2 are L.I (how Ex.?)

So, our basis is $\{u_1, u_2\} = \{(-1,1,0,0,0), (-1,0,-1,0,1)\}$

$\dim W = \boxed{2}$ #

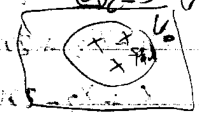
basis عنصر

S spane V

Theorem (5.4.4) let S be a nonempty set of vectors in a vector, space V, then:-

غير فارغة

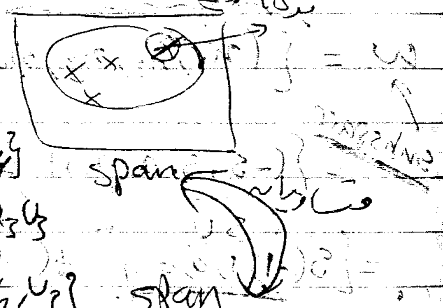
ⓐ If S is a linearly independent set, and $v \in V$ that outside spanne S then the set $S \cup \{v\}$ is still still linearly independent.



ⓐ لو عندي عنصر V مش خارج span لو حطيته ليها يكون تظل مستقل

b) If v is a vector in S that is expressible as linear combination of other elements in S then $S - \{v\}$ span the same space spanned that $\text{span}(S) = \text{span}(S - \{v\})$

ⓑ



$S = \{v_1, v_2, v_3, v_4\}$ span

$v_4 = a_1 v_1 + a_2 v_2 + a_3 v_3$

$S - \{v_4\} = \{v_1, v_2, v_3\}$ span

Ⓐ لو أضفت عنصر لـ linear I يبقى (L.I)

Ⓑ لو شيلت عنصر من span يبقى نفس الـ span

Theorem (5.4.5) If V is an dimesional vector space and if S is a set in V with exactly n vectors then S is a basis for V if either S span V or S is linearly independent.

Proof:- let S be subset of V with n-element ($\dim V = n$). Assume that S is linearly independent set. Assume S is does not span V. That mean their exsist $v \in V$ $\ni v$ is a linear combination of the element in S.

$\Rightarrow S \cup \{v\}$ is a linear independent set condradection since $S \cup \{v\}$ has $n+1$ element, so it is linear dependent.

الجزء الثاني

The other part Exeercise.

Excersise: Show that $v_1 = (-3, 7)$, $v_2 = (5, 5)$ form a basis for R^2.

② عدد المتجهات

② البعد

Show that $\{v_1, v_2\}$ is L.I.

$k_1 v_1 + k_2 v_2 = 0$

$-3k_1 + 5k_2 = 0$

$7k_1 + 5k_2 = 0$

$-10k_1 = 0$

$k_1 = 0$ then $k_2 = 0$

$k_1 = k_2 = 0$

then $\{v_1, v_2\}$ Linearly Independ

بالتالي linearly indep

2) Show that
$v_1 = (2, 0, -1)$, $v_2 = (4, 0, 7)$
$v_3 = (-1, 1, 4)$ form a basis for R^3.
(hint: show that $\{v_1, v_2, v_3\}$ span R^3.
let (x, y, y) &
$a_1 v_1 + a_2 v_2 + a_3 v_3 = (x, y, z)$.
$2a_1 + 4a_2 - a_3 = x$
$a_3 = y$
$-a_1 + a_2 + 4a_3 = z$

$$\begin{bmatrix} 2 & 4 & -1 & x \\ 0 & 0 & 1 & y \\ & & & z \end{bmatrix}$$

spnnd
then

span
consistan — yes
inconsistan — No

Theorem
(5.4.6)

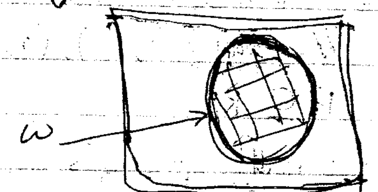
Theorem (5.4.7) مهم **

If W is a subspace of a vector space V which is finite dimensional, then

$\dim W \le \dim V$ ← vector space

If $\dim W = \dim V$, then $W \neq V$

$W \neq V$



$\dim W \le \dim V$

section 5 – Row space, Column space and null space:–

Def:– If A is $m \times n$ matrix, then the subspace of R^n spanned by the row vectors of A is called the row space. Also the subspace of R^n spanned by column vectors is called the column space.
The Solution space of the system $AX = 0$, which a subspace of R^n is called the null space of A.

$AX = 0$

Theorem 5.5.1

A system of A linear equations $AX=b$ is consistant iff b is in the column space of A.

EX:

1) $A = \begin{bmatrix} 2 & 0 & 1 & 5 & 7 \\ 3 & 4 & -2 & 13 & 9 \\ 1 & 2 & 5 & 0 & 12 \end{bmatrix}$

The row space is $w = \text{span}\{(2,0,1,5,7),(3,4,-2,13,9),(1,2,5,0,12)\}$

$w \subseteq R^5$

The colum space is

$H = \text{span}\{(2,3,1),(0,4,2),(1,-2,5),(5,13,0),(7,9,12)\}$

$H \subseteq R^3$

The null space is the solution space of $AX=0$

$$\begin{bmatrix} 0 & 0 & 25 & -21 & 37 \\ 0 & -2 & -17 & 13 & -27 \\ 1 & 2 & 5 & 0 & 12 \end{bmatrix}$$

$25x_3 - 21x_4 + 37x_5 = 0$

$x_5 = t, \quad x_4 = s$

$x_3 = \dfrac{21s - 37t}{25}$

$x_2 = -13s + 27t + \dfrac{1}{25}(21s - 37t)$

$(x_1, x_2, x_3, x_4, x_5) = sv_1 + tv_2$

null space $= \text{span}\{v_1, v_2\}$

Example:

(1) $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is consistant since

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

(2) $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

clearly $a_1, a_2, a_3 \; \exists$

$$\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = a_1\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + a_2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + a_3\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

inconsistant

Theorem (5.5.2): If x_0 denoted any single solution of a consistant linear system $AX=b$, If $v_1, v_2, v_3, \ldots, v_k$ form a basis of null space of A, then that every solution of $Ax=b$ can be express in the form $X = x_0 + c_1v_1 + c_2v_2 + \cdots + c_kv_k$ and conversly for all choice of scalars $c_1, c_2, \ldots, c_k$ the vector x is ...

Ex: Solve the system:

$x_1 + 3x_2 - 2x_3 + 2x_5 = 0$

$2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1$

$5x_3 + 10x_4 + 15x_6 = 5$

$2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 6$

Sol:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -3r-4s-2t \\ r \\ -2s \\ s \\ t \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{3} \end{bmatrix} + r\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$= x_0 + rV_1 + sV_2 + tV_3$

Clearly, x_0 is a partial solution of $AX=b$ and $\{v_1, v_2, v_3\}$ form a basis for null space of the matrix of A.

Bases for Row, Column and null space :-

Theorem (5.5.3) مهم

Elementary row operation doesn't change the null space of a matrix. نول سبيس ما بتتغير بـ ERO

Theorem (5.5.4) مهم : Elementary row operation doesn't change the row space of a matrix

Remark:-

Elementary row operation may change the Column space of a matrix.

Example:

If $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

$A \xrightarrow{R_2 - 2R_1} B$

$(1,2) \in$ Column space of A.

$(1,2) \notin$ Column space of B.

C.S of A $\neq$ C.S of B

EX $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

C.S of A = C.S of B.

Theorem (5.5.5)

مصفوفتين بقدر تحصل على وحدة من التانية بـ R.O.

If A and B are Row equavellent

a) a given set of column ~~space~~ vectors of A is linearly independent $\Leftrightarrow$ iff the corresponding Column vector of B are linearly independent

b) a given set of Colum vectors of A form a basis for the column space of A iff the corresponding column vectors of B for a basis for the column space of B.

توضيح

$$A = \begin{bmatrix} \cdots & C_A^1 & C_A^3 & C_A^{10} & C_A^{100} & \cdots \end{bmatrix} , \quad B = \begin{bmatrix} \cdots & C_B^1 & C_B^3 & C_B^{10} & C_B^{100} & \cdots \end{bmatrix}$$

$\{C_A^1, C_A^3, C_A^{10}, C_A^{100}\}$ L.I

$\Updownarrow$

$\{C_B^1, C_B^3, C_B^{10}, C_B^{100}\}$ are L.I

Theorem (5.5.6) If a matrix R is in REf then the row vectors with leading 1's form a basis for the row space
And the column vector with leading one's form a basis for the column space of R.

كو جئنا لنظرية

$R \rightarrow REf$

$$\begin{matrix} 1 & 0 & 0 & \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

Example :-

$$A = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The Basis for the R.S. is

$(1,-2,5,0,3),(0,1,3,0,0),(0,0,0,1,0)\}$

The Basis for the C.S. is
is

$[(1,0,0,0),(-2,1,0,0),(0,0,1,0)]$

(A) في Row نفسهم أما في Column كله من

2) Find a basis for the row space and Column space ~~and~~ null space of :-

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

Sol: The R.E.f of A is

$$\begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

① For row space the basis

$[(1,-3,4,-2,5,4),(0,0,1,3,-2,-6),(0,0,0,0,1,5)]$

نعبرنا من المقابلات.

② For the Column space the basis is

في Column أختار لي على قبالهم على طول

$\{(1,2,2,-1),(4,9,9,-4),(5,8,9,-5)\}$

يعني باختصار نأخذ الي بقابلهن في Column.

③ null space $\boxed{AX=0}$

$$x_1 + 3x_2 + 4x_3 - 2x_4 + 5x_5 + 4x_6 = 0$$
$$x_3 + 3x_4 - 2x_5 - 6x_6 = 0$$
$$x_5 + 5x_6 = 0$$

تابع ↓

$x_6 = t \quad \Rightarrow x_5 = -5t$

$x_4 = s \quad \Rightarrow x_3 = -3s - 4t$

$x_2 = r \quad \Rightarrow x_1 = 3r + 14s + 37t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 37 \\ 0 \\ -4 \\ 0 \\ -5 \\ 1 \end{bmatrix} + s \begin{bmatrix} 14 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

basis is $\{(3,1,0,0,0,0), (37,0,-4,0,-5,1), (14,0,-3,1,0,0)\}$

مهم جدا

Ex:- ركز على صيغة السؤال . ركز على صيغة السؤال

Find a basis for the ~~vector~~ space spanned by the vectors:- (يسمى basis)

$v_1 = (1,-2,0,0,3)$, $v_2 = (2,-5,-3,-2,6)$

$v_3 = (0,5,15,10,0)$, $v_4 = (2,6,18,8,6)$

Sol.

[1] We put these vectors as a row vector in a matrix it A.

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

[2] We do some elementary row operation to get the REF of A.

$$R = \begin{bmatrix} 1 & -2 & 0 & 0 & 2 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

[3] the vector $\{(1,-2,0,0,2), (0,1,3,2,0), (0,0,1,1,0)\}$ are basis for the row space of R

= row space of A.

= space spanned by $S = \{v_1, v_2, v_3, v_4\}$

So, our basis is

$\{(1,-2,0,0,2), (0,1,3,2,0), (0,0,1,1,0)\}$

مهم جبر ٢
د. عاصفة الفوادي

Example:- Find a basis for the space spanned by the vector in the last example Consisting of these vectors.

بدي basis عناصره من نفس الـ vectors

يعني منشيل من الأربعة بعمله basis

Sol:

$\boxed{1}$ We construct A as before

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

$\boxed{2}$ We Calculate A^T

$$A^T = \begin{bmatrix} 1 & 2 & 0 & 2 \\ -2 & -5 & 5 & 6 \\ 0 & -3 & 15 & 18 \\ 0 & -2 & 10 & 8 \\ 3 & 6 & 0 & 6 \end{bmatrix}$$

$\boxed{3}$ We ~~...~~ Convert A^T to REF وشيء فقط

$$R = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -5 & -16 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

من الـ columns تاعت المقابل للـ

$\boxed{4}$ The basis colum space is.

$\{(1,-2,0,0,3), (2,-5,-3,-2,6), (2,6,18,8,6)\}$

Exercise:- $v_1 = (1,-2,0,3)$, $v_2 = (2,-5,3,6)$, $v_3 = (0,1,3,0)$, $v_4 = (2,-1,4,-7)$, $v_5 = (5,-8,1,2)$

Using two methods. (وأعمل بطريقتين)

sec(6) Rank and nullity:-

Remark:- Consider the matrix A and its transpose A^T, then

(1) The column space of A^T is the row space of A.

(2) The row space of A^T is the column space of A. عكس على طول مش نفس الشي

(3) The null space of A is not always the nullspace of A^T.

Theorem (5.6.1):-

If A is any matrix, then the row space of A and the column space of A have the $\boxed{\text{Same dimention}}$.

المشترك ، البعد

def.: The common dimension of the row space and column space ~~denoted by~~ is called the Rank of A, we can write rank(A).

ملاحظة / rank هو dim لـ row و الـ colum .

The dimension of the null space of A is called the nullity of A and we write nullity(A).

Example:- Find the rank and nullity of

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

Sol:-

$$R = \begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Base

rank (A) = 2 عدد القاعدة

afftex solution, we have AX=0 نعوض

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r\begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

nullity (A) = 4 ← عدد parameters

parameter عدد

Theorem: (5.6.2)

If A is any matrix then $\text{rank}(A) = \text{rank}(A^T)$

Proof:-

$\text{rank}(A) = \dim(\text{Row space of } A) = \dim(\text{column space of } A)$

$\Rightarrow \text{rank}(A) = \text{rank}(A^T)$

Theorem (5.6.3) rank + nullity = عدد الأعمدة

If A is any matrix with n columns then

$\text{rank}(A) + \text{nullity}(A) = n$

Theorem (5.6.4) If A is $m \times n$ matrix, then

(a) rank(A) = the number of leading variables in the solution of AX=0

(b) nullity (A) = the number of the varibles (parameter) in the general Solution of AX=0.

nullity = عدد البارامتر number of Para.

Ex

A^T 7×5

Example:- If A is 5×7 matrix, with rank$(A) = 3$, then

rank + null = ...

(1) nullity $= 7-3 = \underline{4}$ 3

(2) rank$(A^T) = 3 =$ rank(A)

(3) nullity $(A^T) = 5 -$ ran$(A^T) = 5-3 = 2$

Remark:-

For any (every) matrix we have:-

rank$(A) \leq \min(n, m)$

Example:- If A is 3×5 matrix, the maximum of the rank.

$$\text{rank}(A) \leq \min(3,5)$$

$$\text{rank}(A) \leq 3$$

The End of sec (6)

The End of the chapter.

تطبيق

Chapter (7)

Eigenvalues , Eigenvectors:-

section (1):- Eigenvalues and Eigenvectors:-

def:-

If A is ~~any~~ $n\times n$ matrix, then a nonzero vector $X \in R^n$ is called an Eigenvector of A if there ~~exist~~ $\boxed{AX = \lambda X}$ for some $\lambda \in R$

The scalar λ is called an eigenvalue and X is said to be an eigenvector of A Corresponding to λ.

Example:- The vector $\begin{bmatrix}1\\2\end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix}3 & 0\\8 & -1\end{bmatrix}$ corresponding to the eigen value $\lambda = 3$ since

$$\begin{bmatrix}3 & 0\\8 & -1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}3\\6\end{bmatrix} = 3\begin{bmatrix}1\\2\end{bmatrix}$$

eigen vector

$AX = 3X$

$\boxed{\lambda = 3}$

X engine vector

λ engine value

How to Find the eigen Value of matrix A??

1) We write first $AX = \lambda X$

2) since we search a non zero ~~[illegible]~~ vector X that
Solve $AX = \lambda X$ we must have
$\det(\lambda I - A) = 0$
This called the characteristic equation.

تذكير
$AX = \lambda X$
$(\lambda I - A)X = 0$

3) The value of $\det(\lambda I - A)$ is always a polynomial P of λ which called the charactaristing polynomial
That $\det(\lambda I - A) = 0$

$$P(\lambda) = \lambda^n + c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \dots + c_n = 0$$

4) we must find the roots of this polynomial to find the eigen values.

Ex:- Find the eigen values of

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

Sol ⟹

Sol:-
$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix} X = \lambda X$$

The charactaristic equation

$$\det(\lambda I - A) = 0$$

$$\det\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix} \right) = 0$$

$$\begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda - 8 \end{vmatrix} = 0$$

هكذا الحد … نسب الحد

نحصل $\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$

* يأسكتوا وبأخوف تونبتو

So, $\lambda = 4$ is root
$(\lambda - 4)$ is factor

$$\begin{array}{r} \lambda^2 - 4\lambda + 1 \\ \lambda - 4 \,\big)\, \overline{\lambda^3 - 8\lambda^2 + 17\lambda - 4} \\ \lambda^3 - 4\lambda^2 \\ \hline -4\lambda^2 + 17\lambda - 4 \\ -4\lambda^2 + 16\lambda \\ \hline \lambda - 4 \\ \lambda - 4 \\ \hline 0 \quad 0 \end{array}$$

$(x - 4)(x^2 - 4\lambda + 1) = 0$
using the general low we have

The eigen Values are
$\lambda_1 = 4$
$\lambda_2 = 2 - \sqrt{3}$
$\lambda_3 = 2 + \sqrt{3}$

Theorem (7.1.1)

If A is A tringular, then the eigen Values of A are entries in main diagonal.

Example:-

The eigen Values of

$$A = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 5 & 0 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix} \end{array} \qquad A = \begin{bmatrix} \frac{1}{2} & a & 0 \\ -1 & \frac{2}{3} & 0 \\ 5 & p & -\frac{1}{4} \end{bmatrix}$$

$\lambda_1 = \frac{1}{2} \quad \lambda_2 = \frac{2}{3} \quad \lambda_3 = -\frac{1}{4}$

ناخذ عناصر القطر الرئيسي 1, 5, 8, 10.

Theorem (7.1.2) :-

If A is $n \times n$ matrix and λ is a real number then the following statement are ~~equation~~ equivalent

ⓐ λ is eigen value of A.

ⓑ The system $(\lambda I - A)X = 0$ has non trivials

ⓒ There is anon zero Vector X in R^n. such that $AX = \lambda X$

ⓓ λ is a Solution of $\det(\lambda I - A) = 0$

How to find the eigenvectors of A?

(1) First we calculate the eigen values of A.

(2) For each eigen value λ, we Solve the system $(\lambda I - A)X = 0$

(3) The eigen vectors corresponding to λ are the Solution of $(\lambda I - A)X = 0$, that is the null space of $(\lambda I - A)$.

(4) we call this solution the eigenspace of A corresponding to λ.

Example:- Find basis for the eigenspace of :

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \qquad \begin{vmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{vmatrix}$$

Sol: The charctaristicg equation is

$\det(\lambda I - A) = 0 \implies$

$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

امسك الحد الأخير
ما نشتغل على قواسم (4)

to factorize the Polynomial we will substitute by $\pm 1, \pm 2, \pm 4$

$\lambda = 1$ is aroot, so $(\lambda - 1)$ is a factor.

Now, using long division, we have

$$\begin{array}{r|l} & \lambda^2-4\lambda+4 \\ \hline \lambda-1 & \lambda^3-5\lambda^2+8\lambda-4 \\ & \mp\lambda^3\pm\lambda^2 \\ \hline & -4\lambda^2+8\lambda-4 \\ & -4\lambda^2+4\lambda \\ \hline & 4\lambda-4 \\ & -4\lambda\pm4 \\ \hline & 0 \quad 0 \end{array}$$

$(\lambda-1)(\lambda^2-4\lambda+4)=0$

$(\lambda-1)(\lambda-2)(\lambda-2)=0$

$(\lambda-1)(\lambda-2)^2=0$

The eigen values are $\lambda_1=1$, $\lambda_2=2$

① For $\lambda_1=1$

$(\lambda_1 I-A)X=0$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}0&0&-2\\1&2&1\\1&0&3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}1&0&2\\-1&-1&-1\\-1&0&-2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}1&0&2\\-1&-1&-1\\-1&0&-2\end{bmatrix}\xrightarrow{E.R.O}\begin{bmatrix}1&0&2\\0&1&-1\\0&0&0\end{bmatrix}$$

حيكون سؤال سريع يعني بالاجوبة

$z=t$, $y=t$, $x=-2t$

The eigenspace corresponding to $\lambda_1=1$ is

$W_1=\{(x,y,z):(-2t,t,t)\}$

$=\{t(-2,1,1): t\in R\}$

$=\text{span}\{(-2,1,1)\}$ withe basis is

$S_1=\{(-2,1,1)\}$

② For $\lambda_2=2$

$(\lambda_2 I-A)X=0$

$$\begin{bmatrix}2&0&2\\-1&0&-1\\-1&0&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$x+z=0$

$$\begin{pmatrix}2&0&2\\-1&0&-1\\-1&0&-1\end{pmatrix}\xrightarrow{E.R.O}\begin{bmatrix}1&0&1\\0&0&0\\0&0&0\end{bmatrix}$$

$z=t$, $\underline{\underline{y=S}}$, $x=-t$ $\quad = t\begin{bmatrix}-1\\0\\1\end{bmatrix}+s\begin{bmatrix}0\\1\\0\end{bmatrix}$

[illegible]

$W_2=\{(x,y,z):(-t,S,t)\}$

$=\{t(-1,0,1)+s(0,1,0)$ is $t\in R\}$

$=\text{span}\{(-1,0,1),(0,1,0)\}$ with basis

$S_2=\{(-1,0,1),(0,1,0)\}$

Theorem (7.1.3)

If k is a positive integer, λ is ~~an~~ an eigen value of A and X is one of its corresponding eigen vectors then λ^k is an eigen value of A^k and X is one of its corresponding eigen vectors.

$\lambda \quad A \quad X \Rightarrow AX = \lambda X$

$\lambda^k \leftarrow A^k \quad X \Rightarrow A^k X = \lambda^k X$

$$AX = \lambda X$$
$$A(AX) = A(\lambda X)$$
$$A^2 X = \lambda (AX)$$
$$= \lambda(\lambda X)$$
$$= \lambda^2 X$$

Example:-

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}, \text{ with}$$

$\lambda_1 = 1$, $\lambda_2 = 2$ are eigen value.

then A^{10} has eigen values

$\lambda_1 = 1^{10} = 1$, $\lambda_2 = 2^{10} = 1024$

Theorem (7.1.4)

A square matrix A is invertable iff $\lambda = 0$ is not an eigen value of A.

Proof:- $\Rightarrow$ Asume A is $n \times n$ invertable matrix.

Assume that $\lambda = 0$ is an eigen value of A.

$$\det(0I - A) = 0$$
$$\det(-A) = 0$$
$$(-1)^n \det(A) = 0 \qquad \text{لتناقض}$$
$$\det(A) = 0 \qquad \text{contradiction}$$

So, $\lambda = 0$ is not eigen value of A.

$\Leftarrow$ assume $\lambda = 0$ is not eigen value of A.

That mean

$\det(\lambda I - A) \neq 0$

$\Rightarrow \lambda^n + c_1 \lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n = 0$

then $c_a \neq 0$ [illegible]

Now

$$\det(A) = (-1)^n (-1)^n \det(A)$$
$$= (-1)^n \det(-A)$$
$$= (-1)^n \det(0I - A) = 0$$
$$= (-1)^n \det(\lambda I - A), \lambda = 0$$
$$= (-1)^n [\lambda^n + c_1 \lambda^{n-1} \cdots + c_n], \lambda = (-1)^n c_n \neq 0 \quad A \text{ is invertable}$$

جعل المصفوفة مصفوفة قطرية

Section (2) Diagonalization التقطير

قابلة للتقطير

def: A squere matrix A is called diagonalizable if there is an invertable matrix P such that $P^{-1}AP$ is diagonal matrix. The matrix P is said to diagonalize A. (تقطر A)

Theorem 7.2.1 If A is nxn matrix then the following are equivellent

الجمل التالية متكافئة

a) A is diagonalizable.

b) A has n linearly independent eigen vector.

كيف نجعلها مصفوفة قطرية

?? How to diagonalize a matrix A :- ??

خطوات الحل

① We calculate the Eigen values of A.

② We find for each eigen value λ of A, a basis for the corresponding eigen space.

③ We collect these vector (basis). نجمع

\# vectors < (n) رتبة المصفوفة

④ If the Number of these vector are less than n, then A is not diagonalizable.

لا يمكن تقطيرها

$A_{15\times15}$

(2) 10 not dia. × 13 < 15 $\boxed{15 = 15}$

⑤ Other wise, by early ! theorem if these vectors number are n, then they are linearly independent.

⑥ We construct the matrix P by putting these vectors as columns. نضعهم كأعمدة في مصفوفة

⑦ The matrix $P^{-1}AP$ will be diagonal matrix wich its main diagonal entries are the eigen values of A in some other.

Remark:- If P exsist, then it is not uniqe.

معناها لو موجودة فهي ليست وحيدة

Example:- Find the matrix (P) that diagonalizes

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Sol.

The eigen values and there corresponding basis are found later as

$\lambda_1 = 1 \quad S = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 2 \quad S = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

P

مادام عددهم (3) فهو L.I و P موجودة تعال نكتب P .

نغيرهم بأي أعمدة

Now $P = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$P^{-1} = \begin{bmatrix} -1 & 0 & -1 \\ -3 & -4 & -2 \\ 1 & 1 & 1 \end{bmatrix}$ $\qquad \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

So, $P^{-1}AP = \begin{bmatrix} -1 & 0 & -1 \\ -3 & -4 & -2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

للعرض $= \begin{bmatrix} \overset{\lambda_1}{1} & 0 & 0 \\ 0 & \overset{\lambda_2}{2} & 0 \\ 0 & 0 & \overset{\lambda_3}{2} \end{bmatrix}$

لاحظ الناتج diag. عناصر القطر الرئيسي هي الايجنات

يعني لو كان الترتيب

$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Example:- The matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$ is not diagonalizable since its eigen values with there ~~basis~~ eigenspaces basis are

$\lambda_1 = 1 \Rightarrow P_1 = \begin{bmatrix} 1/8 \\ -1/8 \\ 1 \end{bmatrix} \circledast$

$\lambda_2 = 2 \Rightarrow P_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \circledast$

عددهم (2) فهو $2 < n$
then is not

Theorem 7.2.2 :- If $v_1, v_2, \dots, v_k$ are eigen vectors of A corresponding to destinct (مختلف) eigen values $\lambda_1, \lambda_2, \dots, \lambda_k$, then $\{v_1, v_2, \dots, v_k\}$ is linearly independent set.

توضيح $\begin{matrix} \lambda_1 & \lambda_2 & \dots & \lambda_k \\ \downarrow & \downarrow & & \downarrow \\ v_1 & v_2 & & v_k \end{matrix}$

Theorem 7.2.3 If nxn matrix A has distinct (مختلف) eigen values, then it is diagonalizable

اذا اختلفت عدد مختلف ← can ← diag

Example:- The matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$ is diagonalizable since it has three distinct eigen values $\lambda_1 = 4$, $\lambda_2 = 2 + \sqrt{3}$, $\lambda_3 = 2 - \sqrt{3}$

مختلفات ثلاثة

Example:- The matrix $A = \begin{bmatrix} -1 & 2 & 4 & 0 \\ 0 & 3 & 1 & 7 \\ 0 & 0 & 5 & 8 \\ 0 & 0 & 0 & -2 \end{bmatrix}$ is diganalizable with eigen values

$\lambda_1 = -1$, $\lambda_2 = 3$, $\lambda_3 = 5$, $\lambda_4 = -2$

diagonalizable يعني ← مختلفات

Computing Powers of matrix:-

This method is valued only for diagonal matrices:-

The method is as follow:-

① If A is diagonalizable, then $\exists$ matrix $P \ni P^{-1}AP = D$ (diagonal)

② Now, $(P^{-1}AP)^2 = D^2$

$$P^{-1}APP^{-1}AP = D^2$$

$$P^{-1}A^2P = D^2$$

by indication, we can say every positive integer k, we have

$$\therefore P^{-1}A^kP = D^k$$

So $\boxed{A^k = PD^kP^{-1}}$

③ If $D = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ 0 & \cdots & & \cdots & d_n \end{bmatrix}$

$$D^k = \begin{bmatrix} d_1^k & 0 & 0 & 0 & \cdots & 0 \\ 0 & d_2^k & 0 & 0 & \cdots & 0 \\ 0 & 0 & d_3^k & 0 & \cdots & 0 \\ 0 & 0 & 0 & & & d_n^k \end{bmatrix}$$

تعطي diagonizal

So, we can calculate A^k trivialy by this method.

Ex:- find A^{13} if $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

Sol.

we know that $P = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

and $P^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ -3 & -4 & -2 \\ 1 & 1 & 1 \end{bmatrix}$

$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ now $A^{13} = PD^{13}P^{-1}$

$$A^{13} = P D^{13} P^{-1}$$

$$= \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1^{13} & 0 & 0 \\ 0 & 2^{13} & 0 \\ 0 & 0 & 2^{13} \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -3 & -4 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A^{13} = \begin{bmatrix} -8190 & 0 & -16382 \\ 8191 & 8192 & 8191 \\ 8191 & 0 & 16383 \end{bmatrix}$$

T هو المتغير

Chapter (8) :-

linear Transformations :- التحويلات الخطية

Sec (1) :- General linear transformation :- التحويلات الخطية العامة

def :- If $T : V \longrightarrow W$ (دالة; vector space; vector space) is a function from a vector space (V) to a vector space (W), then T is called a linear transformation (تحويل خطي) from V into W if for all u and v in V and scalars c we have

① $T(u+v) = T(u) + T(v)$ تتوزع على الجمع

② $T(cu) = c\,T(u)$ يطلع الثابت برة (جوش) تأثير خطي

If $\boxed{V = W}$, then T is called a linear operator.

اذا كانت $\boxed{V=W}$ وتحقق الشرطين

Example: ① The zero linear transformation If V and W are two vector spaces, the zero linear tranformation is defined as :-

$O : V \longrightarrow W$ given by

Identity of $O(u) = O_W \quad \forall u \in V$

let $u, v \in V$, a be scalar then :-

$T(u+v) = O_W = O_W + O_W$
$= T(u) + T(v)$
$= O(u) + O(v)$

$O(u+v) = O(u) + O(v)$ تحقق الشرط الاول

$O(au) = O_W = a \cdot O_W$
$= a\,O(u)$

$O(au) = a(O(u))$ تحقق الشرط الثاني

linear
المؤثر المحايد

② The identity operator :-
If V be a vector space define the map نقل
$I : V \longrightarrow V$ given by
by $I(u) = u$ المحايد يعطي العنصر نفسه
If $u, v \in V$, a a scalar, then
1) $I(u+v) = u+v$
$= I(u) + I(v)$

$$\boxed{I(u+v) = I(u) + I(v)}$$

2) $I(au) = au = a\,I(u)$

$$\boxed{I(au) = a\,I(u)}$$

③ Define $T : R^n \longrightarrow R^m$ by
$T(x) = AX \;\; \forall X \in R^n$ where A is a fixed $m \times n$ matrix. This is a linear transformation (It is a type of Eqlidian transformation).
نوع من أنواع

let $x_1, x_2 \in R^n$, a be any scalar.
1) $T(X_1 + X_2) = A(X_1 + X_2)$
$= AX_1 + AX_2$
$T(X_1 + X_2) = T(X_1) + T(X_2)$ ← تحقق الشرط الأول

2) $T(aX_1) = A(aX_1)$
$T(aX_1) = a(AX_1) = aT(X_1)$

مثال عددي
* Numirical Example:
$T : R^2 \to R^3$ given by

$$T(X) = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 5 & 3 \end{bmatrix}$$

$T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)$

$$\cancel{T(1,2)} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 11 \end{bmatrix}$$

④ The relative cordinates linear transformation:-

Given that V is an n-dimentional vector space with basis
$S = \{w_1, w_2, \ldots, w_n\}$
Remember that, the cordinate of a vector $v \in V$ realative to the basis S is given by
$(v)_S = (k_1, k_2, \ldots, k_n)$, where

$$v = k_1 w_1 + k_2 w_2 + \cdots + k_n w_n$$

Now, defined the map
$T : V \to R^n$ given by
$T(u) = (v)_S$. let $u, v \in V$, a be a scalar, then

If $u = k_1 w_1 + k_2 w_2 + \cdots + k_n w_n$
and $v = b_1 w_1 + b_2 w_2 + \cdots + b_n w_n$
then $(u)_s = (k_1, k_2, \cdots, k_n)$

$$(v)_s = (b_1, b_2, \cdots, b_n)$$

$$(u+v)_s = (k_1+b_1, k_2+b_2, \cdots, k_n+b_n)$$

Now:-

$$\begin{aligned} T(u+v) &= (u+v)_s \\ &= (k_1+b_1, k_2+b_2, k_3+b_3, \cdots, k_n+b_n) \\ &= (k_1, k_2 \cdots k_n) + (b_1, b_2, \cdots b_n) \\ &= (u)_s + (v)_s \\ &= T(u) + T(v) \end{aligned}$$

$$\begin{aligned} T(au) &= (au)_s \\ &= (ak_1, ak_2, \cdots, ak_n) \\ &= a(k_1, k_2, \cdots, k_n) \\ &= a(u)_s \\ &= a\,T(u) \end{aligned}$$

So, L.T #

⑤ Define:-

$T : P_n \longrightarrow P_{n+1}$

given by

$T(P(x)) = x\,P(x)$

$T(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) = a_0 x + a_1 x^2 + a_2 x^3 + \cdots + a_n x^{n+1}$ This is a linear transformation, since if $f, g \in P_n$, a be scalar then:-

1) $T(f+g) = x(f+g)$
$= xf + xg$
$= T(f) + T(g)$ تحقق الشرط الأول

2) $T(af) = x(af)$
$= a(xf)$
$= a\,T(f)$ تحقق الشرط الثاني

Numirical Example:-

$T : P_2 \to P_3$, then

$$\begin{aligned} T(1 - 2x + x^2) &= x(1 - 2x + x^2) \\ &= x - 2x^2 + x^3 \end{aligned}$$

⑥ $T: P_n \longrightarrow P_n$ given by

$T(p(x)) = p(ax+b)$

$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

$u, v \in V$

α scalar

$p(ax+b) = a_0 + a_1(ax+b) + a_2(ax+b)^2 + \cdots + a_n(ax+b)^n$

$\therefore$ let $u, v \in V$ & α scalar

$T(u+v) = T($

⑦ $D: C'(\mathbb{R}) \longrightarrow F(\mathbb{R})$

$C'(\mathbb{R})$: The set of all Functions on $\mathbb{R}$ with continous first derivatives. given by:-

$D(f) = f'$

Is a linear transformations since

$D(f+g) = (f+g)'$

$= f' + g' = D(f) + D(g)$

$D(cf) = (cf)'$

$= cf' = c\,D(f)$.

Example

$D(e^x \sin x)$

$= e^x \sin x + e^x \cos x$

⑧ $L: C(\mathbb{R}) \longrightarrow C'(\mathbb{R})$

$C(\mathbb{R})$: The set of all continous function on $\mathbb{R}$ given by:-

$$L(f) = \int_0^x f(t)\,dt$$

Is a linear transformation since

$L(f+g) = \int_0^x (f+g)(t)\,dt$

$= \int_0^x f(t)\,dt + \int_0^x g(t)\,dt$

$= L(f) + L(g)$

$L(cf) = \int_0^x c f(t)\,dt$

$= c\int_0^x f(t)\,dt = c\,L(f)$

مثال

$L(\cos 2x + 5)$

$= \int_0^x (\cos 2t + 5)\,dt$

$= \frac{\sin 2t}{2} + 5t \Big|_0^x$

$= \frac{\sin 2x}{5} + 5x$

(9) $T: R^3 \to R^2$ given by

$$T(x,y,z) = (2x+y-z \,,\, x+y-2z)$$

Is a linear transformation since.

$$T((x,y,z)+(a,b,c))$$
$$= T(x+a, y+b, z+c)$$
$$= (2(x+a)+(y+b)-(z+c),\ x+a+y+b-2(z+c))$$
$$= ((2x+y-z)+(2a+b-c),\ (x+y-2z)+(a+b-2c))$$
$$= (2x+y-z,\ x+y-2z) + (2a+b-c,\ a+b-2c)$$
$$= T(x,y,z) + T(a,b,c)$$

Same thing [illegible]

$$T(k(x,y,z)) = kT(x,y,z)$$

This tranformation each be written as

$$T(x,y,z) = (2x+y-z \,,\, x+y-2z)$$
$$= x(2,1) + y(1,1) + z(-1,-2)$$
$$= \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = AX$$

where $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \end{bmatrix}$

Example (10) : ~~Def~~ $T: M_{m\times n} \to R$ by

$T: M_{m\times n} \to R$ by

$$T(A) = \det(A)$$

is not a linear transformation since

$$T(A+B) \neq T(A)+T(B)$$ In general!

Remark: We can generalize the definition as:- $\forall v_1, v_2, \dots, v_n \in V$

$a_1, a_2, \dots, a_n$ be a scalars, we have

$$T(a_1v_1 + a_2v_2 + \dots + a_nv_n)$$
$$= a_1T(v_1) + a_2T(v_2) + \dots + a_nT(v_n)$$

Theorem (8.1.1)

If $T: V \longrightarrow W$ be a linear transformation then

[illegible]

$\overset{\text{zero } V}{}\quad\overset{\text{zero } W}{}$

a) $T(0_V) = 0_W$

b) $T(-V) = -T(V)$

c) $T(V-W) = T(V) - T(W)$

Proof Excersises ارجع للكتاب

---

let $\cancel{0}$ 0 is any vector in V and

$0V = 0$

$\cancel{0}T(0V) = 0T(V) = \boxed{0}$ ※

$T(-V) = T((-1)V) = (-1)T(V) = -T(V)$

Let ~~T(V-W)~~

$T(V) + T((-1)W)$

$= T(V) + (-1)T(W)$

$= T(V) - T(W)$

※

Finding L.T from Images of Basis Vectors:-

If V is a finite dimenstional vector space with basis $S = \{v_1, v_2, \dots, v_n\}$ and

$T: V \longrightarrow W$ is a linear transformation then:-

we can determined the transformation Completely using the images of the basis vectors $v_1, v_2, \dots, v_n$ as following:-

If $u \in V$, then we can write

$u = c_1v_1 + c_2v_2 + \dots + c_nv_n$

Now

$T(u) = T(c_1v_1 + c_2v_2 + \dots + c_nv_n)$

$= c_1T(v_1) + c_2T(v_2) + \dots + c_nT(v_n)$

That is we only Find the values

$T(v_1), T(v_2), \dots, T(v_n)$

Example: Consider basis $S = \{v_1, v_2, v_3\}$ of R^3

where $v_1 = (1,1,1)$, $v_2 = (1,1,0)$, $v_3 = (1,0,0)$

If $T: R^3 \to R^2$ is a linear transformation

Such that $T(v_1) = (1,0)$, $T(v_2) = (2,-1)$, $T(v_3) = (4,3)$

Find ~~T(2~~ $T(2,-3,5)$

[illegible] ↓

Sol:- First we find the scalars c_1, c_2, c_3 such that for an arbitary element (x,y,z) in $\mathbb{R}^3$.

we have $(x,y,z) = c_1 v_1 + c_2 v_2 + c_3 v_3$

$$= c_1(1,1,1) + c_2(1,1,0) + c_3(1,0,0)$$

بعد الحل

after solution we have

$c_1 = z,\ c_2 = y-z,\ c_3 = x-y$

that

$$(x,y,z) = z(1,1,1) + (y-z)(1,1,0) + (x-y)(1,0,0)$$

So:

$$T(x,y,z) = z(T(1,1,1) + (y-z)(1,1,0) + (x-y)T(1,0,0)$$

$$T(x,y,z) = z(1,0) + (y-z)(2,-1) + (x-y)(4,3)$$

$$T(x,y,z) = (\underbrace{z+2y-2z+4x-4y}, \underbrace{-y+z+3x-3y})$$

$$= (4x-2y-z,\ 3x-4y+z)$$

$T(2,-3,5)$

$$= (8+6-5,\ 6+12+5)$$

$$= (9, 23)$$

$$\boxed{T(X) = \begin{bmatrix} 4 & -2 & -1 \\ 3 & -4 & 1 \end{bmatrix} X}$$

def. If $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations, then the composition of T_2 with T_1 denoted by $T_2 \circ T_1$ (read T_2 circle T_1) is a function defined by $(T_2 \circ T_1)(u) = T_2(T_1(u))$ where $T_2 \circ T_1 : U \to W$

$$\begin{tikzcd} U \arrow[r, "T_1"] \arrow[rr, bend right, "T_2 \circ T_1"'] & V \arrow[r, "T_2"] & W \end{tikzcd}$$

Theorem (8.1.2): If $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations, the composition $T_2 \circ T_1 : U \to W$ is also a linear transformation.

Proof:- let x & $y \in U$, a be scalar, then

$$\to T_2 \circ T_1(x+y) = T_2(T_1(x+y))$$
$$= T_2(T_1(x) + T_1(y))$$
$$= T_2(T_1(x)) + T_2(T_1(y))$$
$$= T_2 \circ T_1(x) + T_2 \circ T_1(y)$$

تحقق شرط

$$\to T_2 \circ T_1(ax) = T_2(T_1(ax)) = T_2(aT(x))$$
$$= aT_2(T(x))$$
$$= aT_2 \circ T_1(x)$$

الشرط الثاني

$T_2 \circ T_1$ is linear transformation

Example: let $T_1 : P_1 \longrightarrow P_2$, $T_2 : P_2 \longrightarrow P_2$ given by

$$T_1(P(x)) = x\,P(x)$$
$$T_2(P(x)) = P(2x+4)$$

Now, $T_2 \circ T_1 : P_1 \longrightarrow P_2$, and given by

$$T_2 \circ T_1(P(x)) = T_2(T_1(P(x))) = T_2[x\,P(x)]$$
$$= (2x+4)\,P(2x+4)$$

$$T_2 \circ T_1(5x-2) = T_2(T_1(5x-2))$$
$$= T_2(x(5x-2)) = T_2(5x^2-2x)$$
$$= 5(2x+4)^2 - 2(2x+4)$$
$$= 20x^2 + 80x + 80 - 4x - 8$$
$$= 20x^2 + 76x + 72$$

Is $T_1 \circ T_2$ define : Note define.

Remark:

1) $T \circ I = I \circ T = T$

2) $(T_3 \circ T_2) \circ T_1 = T_3 \circ (T_2 \circ T_1)$

section (2) :- Kernal and Range :-

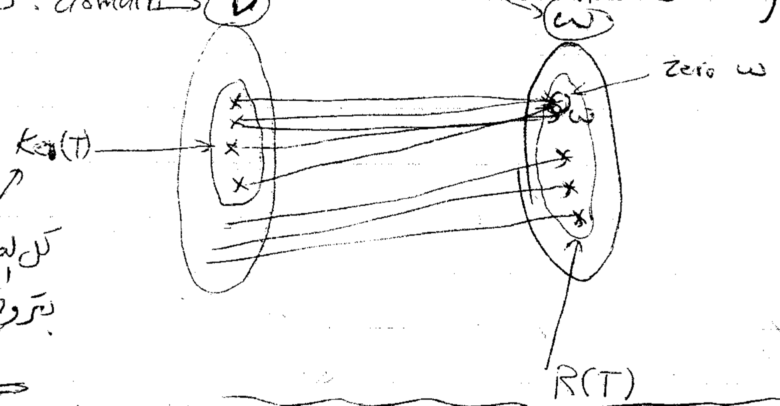
def: If $T : V \to W$ is a linear transformation then the set of vector in V that T maps (تحمل) into 0_W is called the kernal of T denoted by $\ker(T)$

$$\ker T = \{u \in V : T(u) = 0\}$$

The set of all vectors in W that are images (صور) under T. (العناصر التي تحولت لـ T) of at least one vector in V is called the range of T denoted by $R(T)$.

$$R(T) = \{w \in W ;\ w = T(v) \text{ for some } v \in V\}$$

domain مجال → (V) codomain مجال مقابل → (W)



كل العناصر اللي بتروح ع zero w
(0_W)

Example:
$T : R^n \longrightarrow R^m$ where $\boxed{T(x) = Ax}$ and A is $m \times n$ matrix.

$$\ker(T) = \{x \in R^n : T(x) = 0\}$$
$$= \{x \in R^n ; T(x) = Ax = 0\}$$
$$= \{x \in R^n : Ax = 0\}$$

null space

$= $ The null space of A

$R(T) = \{y \in R^m : T(x) = y \text{ for some } x \in R^n\}$

$= \{y \in R^m : Ax = y\}$

$= \{y \in R^m : Ax = y \text{ is consistant}\}$

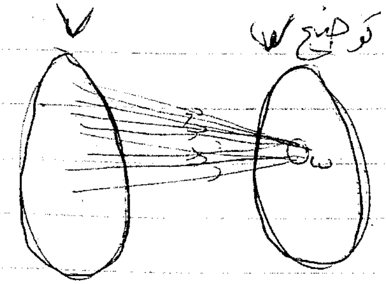
$=$ The colum space of A.

زي

2) $O : V \to \omega$ where $O(u) = 0$ zero $\forall u \in V$

زي

$\ker(O) = \{u \in V : O(u) = 0\} = V$



$R(O) = \{\omega \in W : O(u) = \omega \text{ for some } u \in V\}$

$= \{0\}$

3) $I : V \to V$ where $I(u) = u$

$\ker(I) = \{u \in V : I(u) = 0\}$

$\{u \in V : u = 0$

صفر $= \underline{\{0\}} \Leftarrow$ أصفار

Range: كل العناصر التي في المجال المقابل التي تأتي إليها الأسهم .

$R(I) = \{v \in V : T(u) = v \text{ for some } u \in V\}$

$= \{u \in V : u = v \text{ for some } u \in V\}$ Range الكل

$= V$

(4) $D : C'(R) \to F(R)$ given by

$D(f) = f'$

$\ker(D) = \{f \in C'(R) : Df = 0\}$

$= \{f \in C'(R) : f' = 0\}$

The set of all constant functions.

سير الدوال التي مشتقتها صفر الثوابت .

Theorem (8.2.1): If $T : V \to \omega$ be a linear transformation

a) $\ker(T)$ is a subspace of V.

b) $R(T)$ is a subspace of ω.

مهم Proof:- Ex

ارجع للكتاب

def: If $T: V \to W$ where V, W are vector space, is a linear transformation, then the dimension of the range of T is called the rank of T denoted by rank (T) and the dimension of kernal T is called the nullity of T denoted by nullity (T).

Theorem (8.2.2)

If A is $m\times n$ matrix and $T_A: R^n \to R^m$ is a multiplication by ($T_A(x) = Ax$), then

a) nullity (T_A) = nullity (A)

b) rank (T_A) = rank (A)

Example: If $T_A: R^6 \to R^4$ where

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 3 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

find rank T_A, nullity T_A

rank (T_A) = rank $(A) = 2$

nullity (T_A) = nullity$(A) = 4$

Theorem (8.2.3)

If $T: V \to W$ is a linear transformation from n dimensional vector space of V to W, then

$$\text{rank}(T) + \text{nullity}(T) = n$$

Sec(3):- Invers linear Transformation:-

Def:- a linear transformation $T: V \to W$ is said to be one to one (1-1) if T maps distinct elements in V into distinct vectors in W. That if $x \neq y$ in V then $T(x) \neq T(y)$ in W or:-

$\forall x, y \in V \ni T(x) = T(y)$, we have $x = y$

Ex:- 1) $T: P_n \to P_{n+1}$ given by $T(P(x)) = xP(x)$ is one-to-one linear transformation since:-

Let $P_1, P_2 \in P_n$ such that:

$T(P_1) = T(P_2)$

$x P_1(x) = x P_2(x) \quad \forall (x \in R)$

$x(P_1(x) - P_2(x)) = 0 \quad \forall x \in R$

since x is arbitary

$\forall x \in R \quad x = 0$

False

So T is one to one

$P_1(x) = P_2(x) \quad \forall x \in R$

$P_1 = P_2$

فرق بين فهمها شيء وتذكرها بالحفظ

C1 A is invertable

thus T is 1-1 since

if $T(x_1) = T(x_2)$

$A x_1 = A x_2$

$A^{-1} A x_1 = A^{-1} A x_2$

$x_1 = x_2$

$\Rightarrow T$ is 1-1

C2 A is not inv

If $T(x_1) = T(x_2)$

$A x_1 = A x_2$

$A(x_1 - x_2) = 0$

with non trivial sol

$x_1 - x_2 \neq 0$

$x_1 \neq x_2$

but $T(x_1) = T(x_2)$

So T is not 1-1

Remark: $T(Ax) = Ax$ is 1-1 iff A is inv

مهم جداً جداً

سؤال الامتحان

2) $T : R^n \longrightarrow R^n$ given by $T(x) = AX$ where A is $n \times n$ matrix is one-one iff A is invertable

Sol: since if A is invertable and $x_1, x_2 \in R^n \ni T(x_1) = T(x_2)$ بالاتجاهين

$AX_1 = AX_2$

$X_1 = X_2$

$\Rightarrow T$ is one-one

assume T is one-one. Consider the system $AX = 0$. This can be written as $T(X) = 0$

but $0 = T(0)$

$\Rightarrow T(X) = 0 = T(0)$

but T is one-one

$\Rightarrow X = 0$

So $AX = 0$ has only trivial solution

$\Rightarrow A$ is invertable

3) $D: C'(R) \longrightarrow F(R)$ given by $D(f) = f'$ is not one-one

since $f_1 = 2,\ f_2 = 4 \ni C'(R)$ with $f_1 \neq f_2$

but $D(f_1) = 0 = D(f_2)$

So D is not one-one

Theorem (8.3.1)

If $T: V \to W$ is a linear transformation then the following are equivellent :-

$$\begin{tikzcd}[column sep=small] & a \arrow[dr] & \\ c \arrow[ur] & & b \arrow[ll] \end{tikzcd}$$

a) T is 1-1

b) $\ker(T) = 0$

c) $\text{nullity}(T) = 0$

[illegible]

Proof:-

using the definition, clearly $b \Leftrightarrow c$

so we only proof that $a \Leftrightarrow b$

$a \Rightarrow b$

assume T is one-one L.T.

let $X \in \ker(T) \Rightarrow T(x) = 0$

So, $T(X) = 0 = T(0) \implies X = 0$

So, $\ker(T) = \{0\}$

$\Leftarrow b \Rightarrow a$

assume $\ker(T) = 0$

let $X, y \in V \ni T(X) = T(y)$

$\Rightarrow T(x) - T(y) = 0 \Rightarrow T(X-y) = 0$

$X - y \in \ker(T) = \{0\}$

$X - y = 0$ one-one

$\Rightarrow x = y$ the T is 1-1

$b \Leftrightarrow c$ $\ker T \{0\}$ iff $\dim(\ker T) = \dim\{0\}$ iff nullity $T = 0$

Theorem (8.3.2)

vector space
عدد محدود من

If V is a finite dimensional vector space and $T: V \to V$ is a linear operator then the following are equivellent:-

a) T is one to one. onto اذا كان

b) $\ker(T) = \{0\}$ iff 1-1 [illegible]

c) $\text{nullity}(T) = 0$ [illegible]

d) $R(T) = V$ Range $= V$ [illegible]

Proof:- [illegible]

from theorem (8.3.1) $\implies a \Leftrightarrow b \Leftrightarrow c$

we will prove that $c \Leftrightarrow d$

$c \Rightarrow d$:- assume $\text{nullity}(T) = 0$

So, $\text{rank}(T) = n - \text{nullity}(T) = n - 0 = n$

$\dim(R(V)) = \text{rank}(T) = n = \dim(V)$

$\Rightarrow R(T) = V$

$d \Rightarrow c$ If $R(T) = V$, then

$\text{rank}(T) = \dim(R(T)) = \dim(V) = n$

$\Rightarrow \text{nullity}(T) = n - \text{rank}(T) = n - n = 0$

Example:- $T_A : R^4 \to R^4$ where $T(X) = AX$

and $A = \begin{bmatrix} 1 & 3 & -2 & 4 \\ 2 & 6 & -4 & 8 \\ 3 & 9 & 1 & 5 \\ -1 & 1 & 4 & 8 \end{bmatrix}$

Is not 1-1 since $\det(A) = 0$

So A is not invertable.

---

def:- Given that $T : V \to W$ is one_one map from V into R(T) then we can defined the inverse of T denoted by T^{-1} as a function from R(T) into V as:-

$T^{-1} : R(T) \to V$ given by

$T^{-1}(w) = V$, where

$T(V) = w$

clearly $T^{-1}(T(u)) = V \quad \forall u \in V$

and $T(T^{-1}(w)) = w \quad \forall w \in R(T)$

Example:- ① $T : P_n \to P_{n+1}$ given by $T(P(x)) = xP(x)$ is 1-1. Now, we can define

$T^{-1} : R(T) \to P_n$ as:-

Now, for $f \in R(T)$, we must have

$f = T(P)$ for some $P \in P_n$

$f = T(p(x)) = T(c_0 + c_1x^1 + \cdots + c_nx^n)$

$= c_0X + c_1X^2 + \cdots + c_nX^{n+1}$.

So, $T^{-1}(f) = T^{-1}(c_0X + c_1X^2 + \cdots + c_nX^{n+1})$

$= c_0 + c_1X + \cdots + c_nX^n$

2) If $T : R^3 \to R^3$ given by

$T(X_1, X_2, X_3) = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$

domain عنصر

Find a formula for T^{-1} If it is exsist.

Sol.

Clearly T^{-1} exsist since T is one-one since $\det(A) \neq 0$. Now

$T^{-1} : R(T) \to R^3$, given

$(a, b, c) \in R(T)$

$\Rightarrow (a, b, c) = T(X_1, X_2, X_3)$ for some $(X_1, X_2, X_3) \in R^3$

$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$

عنصر من Rang

صورة عنصر من dom

تحليل (مثال) $\rightarrow$ $\boxed{\begin{array}{c} y = A x \\ x = A^{-1} y \end{array}}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -2 & -3 \\ -11 & 6 & 9 \\ -12 & 7 & 10 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$= (\underbrace{4a - 2b - 3c}_{x_1}, \underbrace{-11a + 6b + 9c}_{x_2}, \underbrace{-12a + 7b + 10c}_{x_3})$$

Theorem (8.3.3)

If $T_1 : U \rightarrow V$, $T_2 : V \rightarrow W$ are one to one linear trasformation then :-

1) $T_2 \circ T_1$ is one-to-one

2) $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$ يتوزع عكس

Proof:-

$T_2 \circ T_1 : U \rightarrow W$

let $x, y \in U \ni$

$T_2 \circ T_1 (x) = T_2 \circ T_1 (y)$

$T_2 (T_1(x)) = T_2 (T_1(y))$

$\Rightarrow T_1(x) = T_1(y)$ (T_2 is one to one)

$x = y$ (T_1 is one to one)

So, $T_2 \circ T_1$ is one-to-one

مهم جداً

b) **Excersise :-** مطلوب من الكتاب

Show $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$

$(T_2 \circ T_1)^{-1}(w) = (T_1^{-1} \circ T_2^{-1})(w)$

$u = (T_2 \circ T_1)^{-1}(w)$

$u = (T_1^{-1} \circ T_2^{-1})\, w$

$w = (T_2 \circ T_1)(u)$

$u = (T_1^{-1} \circ T_2^{-1})(w)$

$(T_2 \circ T_1)\, u = w$

$\boxed{w = w}$ #

sec(4) Matrices of General linear Transformation:-

Given that a linear transformation $T: V \to W$ where V and W and finite dimensional vector spaces, let $B = \{u_1, u_2, \ldots, u_n\}$ be a basis for V and
$B' = \{v_1, v_2, \ldots v_m\}$ be a basis for W.

Then T can be regarded as a matrix transformation as following:-

1) Given that $x \in V$ then let $[x]_B$ be that the coordinate vector of x relative to B, we know that $[x]_B \in \mathbb{R}^n$.
and $[T(x)]_{B'}$ be the coordinate vector of $T(x)$ relative to B', we know that $[T(x)]_{B'} \in \mathbb{R}^m$

$$\begin{tikzcd} V & W \\ x \arrow[r, "T"] \arrow[d] & T(x) \arrow[d] \\ {[x]_B} \arrow[d] & {[T(x)]_{B'}} \arrow[d] \\ \mathbb{R}^n \arrow[r, "{[T]_{B'B}}"] & \mathbb{R}^m \end{tikzcd}$$

we must find the matrix A such that
$A[x]_B = [T(x)]_{B'}$

A is called the matrix for T with respect to the basis B' and B.

After calculations, we find that the column of A are the vectors
$[T(u_1)]_{B'}, [T(u_2)]_{B'} \ldots [T(u_n)]_{B'}$

$$u_1 \longrightarrow T(u_1) \longrightarrow [T(u_1)]_{B'}$$
$$u_2 \longrightarrow T(u_2) \longrightarrow [T(u_2)]_{B'}$$
$$\vdots$$
$$u_n \longrightarrow T(u_n) \longrightarrow [T(u_n)]_{B'}$$

So, that $A = [[T(u_1)]_{B'}, [T(u_2)]_{B'}, [T(u_3)]_{B'}, \ldots [T(u_n)]_{B'}]$

3) The $m \times n$ matrix A will be denoted by $[T]_{B',B}$ and we have the formula

$$\boxed{[T]_{B',B}[X]_B = [T(x)]_{B'}}$$

$V=W$

4) If $T: V \longrightarrow W$ is a linear operator, we will write $[T]_B$ which called the matrix T with respect to basis B and write

$$\boxed{[T]_B [X]_B = [T(x)]_B}$$

Ex.

1) If $T: P_1 \longrightarrow P_2$ given by

$T(p(x)) = x\,P(x)$

let $B = \{u_1, u_2\} = \{1, x\}$ be a basis for P_1
and $B' = \{v_1, v_2, v_3\} = \{1, x, x^2\}$ be a basis for P_2
to find $[T]_{B',B}$ we will do the following:

① Find the values of $T(u_1), T(u_2), \ldots$, etc.

$T(1) = x.1 = x$

$T(x) = x.x = x^2$

② we find the coordinate vector of each $T(u_1)$, $T(u_2), \ldots$ with respect to B'

$T(1) = x = 0(1) + 1(x) + 0(x^2)$

$[T(1)]_{B'} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$T(x) = x^2 = 0(1) + 0(x) + 1(x^2)$

$[T(x)]_{B'} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

③ Construct $[T]_{B',B}$

$[T]_{B',B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

we know prove the fact ← verify.

$[T]_{B',B} [X]_B = [T(x)]_{B'}$

let $a + bx \in P_1$

$a + bx = a(1) + b(x)$

$[a+bx]_B = \begin{bmatrix} a \\ b \end{bmatrix}$

$T(a+bx) = ax + bx^2 = 0(1) + a(x) + b(x^2)$

$T(a+bx]_{B'} = \begin{bmatrix} 0 \\ a \\ b \end{bmatrix}$, Now

$T_{B',B}\ [X]_B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ b \end{bmatrix}$

$[T(x)]_{B'}$

Example (2) let $T: R^2 \longrightarrow R^3$ given by

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -5x+13y \\ -7x+16y \end{bmatrix}$$

$$B = \left[\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix}\right], \quad B' = \left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\right\}$$

$$T\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix} = (1)\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 0\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + (-2)\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

نجيب قيمة T على عناصر B

$$\left[T\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right)\right]_{B'} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

$$\left[T\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix}\right)\right] = 3\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + (1)\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + (-1)\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$\left[T\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix}\right)\right]_{B'} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

$$[T]_{B',B} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}$$

Excersise (3)

$T: R^2 \longrightarrow R^2$ given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ -2x+4y \end{bmatrix}$$

$$B = \left[\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right]$$

Find $[T]_B$

$$T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$T\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$\boxed{T_B = \begin{bmatrix} 2 & 3 \\ 2 & 6 \end{bmatrix}}$$

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$T\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$[T]_B = \{(\ )(\ )\}$$

عشان LO ما يزعل منى [illegible]

Ex: $I: V \to V \Rightarrow [I]_B = I$

Ex: $T: R^2 \to R^2$ given by
$T(x,y) = (x+y, -2x+4y)$
$B = \{(1,1), (1,2)\}$ is basis of R^2
Find the matrix of T with respect to B

so $v_1 = (1,1)$ $v_2 = (1,2)$
$w_1 = (1,1)$ $w_2 = (1,2)$
$T(v_1) = (2,2)$
$[T(v_1)]_B = (2,0)$

$T(v_2) = (3,6)$
$[T(v_2)]_B = (0,3)$

$[T]_B = \left[[T(v_1)]_B \; [T(v_2)]_B \right]$
$= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

(2) Use the last matrix to evalute $T(-1,3)$

ok

1) الأحد منتصف
2) الإثنين > جبر
3) الثلاثاء
4) الأربعاء مبادئ
جميعا كم بيلكم
[illegible]

let $u = (-1,3)$
$[u]_B = ??$
$u = av_1 + bv_2$
$(-1,3) = a(1,1) + b(1,2)$
$\left.\begin{matrix} -1 = a+b \\ 3 = a+2b \end{matrix}\right] \begin{matrix} \to b = 4 \\ \to a = -5 \end{matrix}$
$[u]_B = (-5,4)$
Now
$[T]_B [u]_B = [T(u)]_B$
$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -5 \\ 4 \end{bmatrix} = [T(u)]_B$
$[T(u)]_B = \begin{bmatrix} -10 \\ 12 \end{bmatrix}$
$T(u) = -10w_1 + 12w_2$

$= -10(1,1) + 12(1,2)$
$= (2,14)$